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Discrete-time approximation of doubly reflected BSDEs

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Abstract

We study the discrete time approximation of doubly reflected BSDEs in a multidimensional setting. As in Ma and Zhang (2005) or Bouchard and Chassagneux (2006), we introduce the discretely reflected counterpart of these equations. We then provide representation formulae which allow us to obtain new regularity results. We also propose an Euler scheme's type approximation and give new convergence results for both discretely and continuously reflected BSDEs.

Key words: Reflected BSDEs, discrete-time approximation schemes, Game Option, regularity.

MSC Classification (2000): 65C99, 60H35, 60G40.

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1 Introduction

The main motivation of this paper is the discrete time approximation of Backward Stochastic Differential Equations (BSDEs) with two reflecting barriers, also known as doubly reflected BSDEs:

$$\left\{\begin{array}{l} Y_t = g(X_T) + \int_t^T f(X_u, Y_u, Z_u)\mathrm{d}u - \int_t^T (Z_u)'\mathrm{d}W_u + \int_t^T \mathrm{d}K_u^+ - \int_t^T \mathrm{d}K_u^- \\ l(X_t) \le Y_t \le h(X_t)\ ,\ \forall t \in [0,T]\ ,\ a.s. \quad (\mathbf{C}) \\ \int_0^T (Y_s - l(X_s))\mathrm{d}K_s^+ = \int_0^T (Y_s - h(X_s))\mathrm{d}K_s^- = 0\ . \end{array}\right. \tag{1.1}$$

where f, g are Lipschitz-continuous functions, h, l are smooth functions (say C_b^2), and the process X is the solution of a forward SDE

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s\ ,$$

with b and σ Lipschitz-continuous.
These equations can be considered as extensions of simply reflected BSDEs, which are related to optimal stopping problem (American option in finance), see e.g. [10], and whose numerical approximation has been widely studied, see e.g. [2, 3, 5, 15]. Existence and uniqueness of solutions to (1.1) have been first studied by Cvitanic-Karatzas in [7]. There has been a lot of contributions on this subject since then, consisting essentially in weakening the assumptions for the existence of (1.1), see e.g. [1] and the references therein. In economics, [7], among others, shows that these equations are related to stochastic stopping games (Dynkin games) and Ma-Cvitanic [6] connects them to the pricing of Game Options (or Israeli Options), introduced in [12].

In this Markovian setting, [6] shows that the solution of (1.1) is associated to variational inequalities (or obstacles problem) of the type

$$\left\{\begin{array}{l} (u-l) \wedge \{(u-h) \vee -[\partial_t u + b\partial_x u + \frac{1}{2}Tr(\sigma\sigma'\partial_{xx}u) + f(t,x,u,\sigma\partial_x u)]\} = 0 \\ u(T,x) = g(x) \end{array}\right. \tag{1.2}$$

in the sense that $(Y_t, Z_t) = (u(t,X_t), \partial_x u\sigma(t,X_t))$ for $t \in [0,T]$. Thus, studying the discrete time approximation of (1.1) offers alternative numerical methods to estimate the solution of (1.2).

While studying the discrete-time approximation of (1.1), it appeared that the techniques we used, can be applied to a multidimensional setting. Namely, Y takes values in $\mathbb{R}^d$ and each component Y^ℓ verifies:

$$\left\{\begin{array}{l} Y_t^\ell = g^\ell(X_T) + \int_t^T f^\ell(X_u, Y_u, Z_u)\mathrm{d}u - \int_t^T (Z_u^\ell)'\mathrm{d}W_u + \int_t^T \mathrm{d}K_u^{\ell+} - \int_t^T \mathrm{d}K_u^{\ell-} \\ \int_0^T (Y_s^\ell - l^\ell(X_s))\mathrm{d}K_s^{\ell-} = \int_0^T (Y_s^\ell - h^\ell(X_s))\mathrm{d}K_s^{\ell-} = 0\ ,\ \ell \in \{1,\dots,d\}, \end{array}\right. \tag{1.3}$$

and, *almost surely*, for all $t \le T$, Y_t is constrained to take values in the domain $\mathcal{O}_{X_t}$ where

$$\mathcal{O}_x := \{y \in \mathbb{R}^d \mid \forall \ell \in \{1, \ldots, d\},\ l^\ell(x) \le y^\ell \le h^\ell(x)\}.$$

Following [3, 15], we first introduce "discretely reflected" versions of (1.1), meaning that condition (**C**) is imposed only on a deterministic set of times $\Re = \{0 =: r_0 < \ldots < r_\kappa := T\}$:

$$Y_T^{\Re} = \widetilde{Y}_T^{\Re} \quad := \quad g(X_T) \, \in \, \mathcal{O}_{X_T}$$

and, for $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\begin{cases} \widetilde{Y}_t^{\Re} & = & Y_{r_{j+1}}^{\Re} + \int_t^{r_{j+1}} f(X_s, \widetilde{Y}_s^{\Re}, Z_s^{\Re}) \mathrm{d}s - \int_t^{r_{j+1}} (Z_s^{\Re})' \mathrm{d}W_s \ , \\ Y_t^{\Re} & = & \widetilde{Y}_t^{\Re} \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X_t, \widetilde{Y}_t^{\Re}) \mathbf{1}_{\{t \in \Re\}} \ , \end{cases}$$

where $\mathcal{P}(x, y)$ is the projection of $y \in \mathbb{R}^d$ onto $\mathcal{O}_x$.

In the framework of doubly reflected BSDEs, i.e. $d = 1$, this corresponds to stochastic stopping games, where the stopping is allowed only on $\Re \setminus \{T\}$.

We then focus on the discrete-time approximation of such equations. As in [3, 5, 15], we introduce a partition $\pi = \{0 =: t_0 < \cdots < t_n := T\}$ such that $\Re \subset \pi$ and define $(Y^\pi, \bar{Z}^\pi)$ by the backward induction:

$$\begin{cases} \bar{Z}_{t_i}^\pi & = & (t_{i+1} - t_i)^{-1} \, \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})(Y_{t_{i+1}}^\pi)' \mid \mathcal{F}_{t_i}\right] \\ \widetilde{Y}_{t_i}^\pi & = & \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i) f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \\ Y_{t_i}^\pi & = & \widetilde{Y}_{t_i}^\pi \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi) \mathbf{1}_{\{t_i \in \Re\}} \ , \ i \le n-1 \ , \end{cases} \tag{1.4}$$

with terminal condition (recall that $t_n = T$)

$$\widetilde{Y}_T^\pi \quad = \quad Y_T^\pi := g(X_T^\pi) \ .$$

Here, X^π is the Euler scheme associated to X.

As in [3, 5, 15], we show that the error induced by this scheme:

$$\max_{i<n} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - \widetilde{Y}_{t_i}^\pi|^2\Big] \quad + \quad \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_t^{\Re} - \bar{Z}_{t_i}^\pi|^2 \mathrm{d}t\right] \tag{1.5}$$

is intimately related to the regularity of the process $(Y^{\Re}, Z^{\Re})$, or equivalently $(\widetilde{Y}^{\Re}, Z^{\Re})$, through the quantities

$$\max_{i<n} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - \widetilde{Y}_{t_i}^{\Re}|^2\Big] \quad \text{and} \quad \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_t^{\Re} - Z_{t_i}^{\Re}|^2 \mathrm{d}t\right],$$

for which we provide new controls in terms of $|\pi|$, the modulus of π. This is based on a generalization of the representation of $Z^{\Re}$ derived in [3].

In this paper, we essentially rely on the basic concepts developed in [3], but we face two new difficulties:
(i) Contrary to [3] where $\mathcal{O}_x$ is of the form $\{y \in \mathbb{R} : y \geq \psi(x)\}$, we do not have an exact expression of the projected process $\mathcal{P}(X_t, \widetilde{Y}_t^{\Re})$ and the reflection terms are much more intricate to handle.
(ii) In the one dimensional case, a simple Girsanov transformation allows to get rid of the Malliavin derivatives of $Y^{\Re}$ and $Z^{\Re}$ which enter in the representation formula of $Z^{\Re}$ (see section 3). This is no more possible, in general, in our multidimensional setting.
Yet, in the discretely reflected case, we are able to extend the regularity result of [3]. This allows to show that the scheme (1.4) has a convergence rate of at least $|\pi|^{\frac{1}{4}}$. Under stronger regularity conditions on the boundaries and the coefficient of the SDE solved by X, we obtain a convergence rate of at least $|\pi|^{\frac{1}{2}}$ (see section 5.3). Using an approximation argument, we then extend these results to continuously reflected BSDEs. The convergence is obtained under minimal Lipschitz-continuity assumptions with a control of order $|\pi|^{\frac{1}{12}}$. Under stronger regularity conditions, we extend the one dimensional result of [15], but without their uniform ellipticity assumption. Namely, we provide an upper bound of order $|\pi|^{\frac{1}{4}}$ for the approximation error. When the system of BSDEs is decoupled, which is the most important case for financial applications, we improve it to $|\pi|^{\frac{1}{3}}$.

We would like to conclude this introduction by observing that the scheme (1.4) is obviously not directly implementable since it requires the computation of conditional expectations. The global numerical error is then the sum of the discrete time approximation error (1.5) and the numerical error induced by the approximation of the conditional expectations. However, this approximation problem is well understood and [2, 5, 9] among others propose efficient numerical methods, which can be easily adapted to our framework. This paper being already long, we shall not detail this part here and only focus on the discretization error.

The rest of the paper is organized as follows. In Section 2, we define BSDEs which are discretely reflected in a convex domain $\mathcal{O}_x$ of the above form. In Section 3, we provide different representations of $Z^{\Re}$ and use them to study the regularity of $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$ in Section 4. In Section 5, we propose an Euler scheme type approximation of discretely reflected BSDEs and give our main convergence results. Finally, in Section 6, we provide extensions to the continuously reflected case. The Appendix contains the proofs of *a priori* estimates which are used several times in the paper.

Notations: $\mathbb{M}^{n,m}$ is the set of matrix with dimension $n \times m$, we simply write $\mathbb{M}^d$ if $m = n = d$. For $z \in \mathbb{M}^{n,m}$, z^{ij} denotes the (ij) component of z, $z^{i\cdot}$ the i-th row of z, $z^{\cdot j}$ the j-th column and z' its transposed matrix. The space L^p, for $p \geq 1$, is the set of random variables X satisfying $||X||_{L^p} := \mathbb{E}[|X|^p]^{\frac{1}{p}} < \infty$. The norm $|.|$ represents the canonic norm on $\mathbb{R}^d$ or on $\mathbb{M}^d$ and $\langle .,. \rangle$ denotes the usual scalar product on $\mathbb{R}^d$. For a function $f \in C^1$, $\nabla_x f$ denotes the Jacobian Matrix of f with respect to x. Finally, for ease of notations, we shall sometimes write $\mathbb{E}_s[.]$ for $\mathbb{E}\left[. \mid \mathcal{F}_s\right]$, $s \in [0,T]$.

2 Discretely reflected BSDE

2.1 Definition

Let $T > 0$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis supporting a d-dimensional Brownian motion W. We assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by W satisfies the usual assumptions and that $\mathcal{F}_T = \mathcal{F}$.

Let X be the solution on $[0,T]$ of

$$X_t = X_0 + \int_0^t b(X_u)\mathrm{d}u + \int_0^t \sigma(X_u)\mathrm{d}W_u \tag{2.1}$$

where $X_0 \in \mathbb{R}^d$ and $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{M}^d$ satisfy one of the following assumptions, for some positive constant L:

- $(\mathbf{H}x1)$: b, σ are L-Lipschitz continuous.
- $(\mathbf{H}x2)$: b, σ are C^1_b with L-Lipschitz continuous first derivative bounded by L.

Remark 2.1. Observe that, as in [3] and contrary to [15], we make no uniform ellipticity condition on σ. In particular, the standard results of the PDE literature cannot be used to derive strong regularity properties on the solution of the PDE of the form 1.2 associated to 1.3.

Under $(\mathbf{H}x1)$, we clearly have that $X \in \mathcal{S}^2(\mathbb{R}^d)$, where for $p \geq 1$ and $E = \mathbb{R}^d$ or $E = \mathbb{M}^d$, $\mathcal{S}^p(E)$ is the set of E-valued progressively measurable processes U such that $||U||_{\mathcal{S}^p} := ||\sup_{t \in [0,T]} U_t||_{L^p} < \infty$. In particular,

$$||X||_{\mathcal{S}^2} \quad \leq \quad C_L \; , \tag{2.2}$$

where, from now on, C_L denotes a generic constant, whose value may change from line to line, but which only depends on L, T, X_0 and d (we write C^p_L if it also depends on some extra parameter $p \geq 1$).

We then introduce a family of closed convex domains $(\mathcal{O}_x)_{x \in \mathbb{R}^d}$:

$$\mathcal{O}_x \quad := \quad \{y \in \mathbb{R}^d \mid \forall \ell \in \{1, \ldots, d\},\; l^\ell(x) \leq y^\ell \leq h^\ell(x)\} \; , \tag{2.3}$$

where the maps $h, l : \mathbb{R}^d \to \mathbb{R}^d$ satisfy one of the following regularity assumptions:

- $(\mathbf{H}b1)$: h and l are L-Lipschitz continuous.
- $(\mathbf{H}b2)$: for each $\ell \in \{1,\dots,d\}$, h^ℓ and l^ℓ verify for some $(\rho_1^\ell,\rho_2^\ell):\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^d$, $\rho_3^\ell:\mathbb{R}^d\to\mathbb{R}^+$,

$$\begin{aligned}|\rho_1^\ell(x)|+|\rho_2^\ell(x)|+|\rho_3^\ell(x)| &\le L(1+|x|^L)\\ l^\ell(x)-l^\ell(y) &\le \rho_1^\ell(x)'(y-x)+\rho_3^\ell(x)|x-y|^2 \ ,\ \forall\, x,y\in\mathbb{R}^d\ .\\ h^\ell(y)-h^\ell(x) &\le \rho_2^\ell(x)'(y-x)+\rho_3^\ell(x)|x-y|^2 \ ,\ \forall\, x,y\in\mathbb{R}^d\ .\end{aligned}$$

  This assumption is slightly weaker than the semi-convexity assumption of Definition 1 in [2].
- $(\mathbf{H}b3)$: h and l are C_b^2 with L-Lipschitz continuous first and second derivatives bounded by L and there is $\epsilon\in(L^{-1},\infty)$ such that $h^\ell > l^\ell+\epsilon$, for each $\ell\in\{1,\dots,d\}$.

Observe that $(\mathbf{H}b3)\Rightarrow(\mathbf{H}b2)\Rightarrow(\mathbf{H}b1)$.

Given a set of reflection times

$$\Re := \{0=:r_0<r_1<\cdots<r_{\kappa-1}<r_\kappa:=T\}\ ,\ \kappa\ge 1\ ,$$

the solution of the discretely reflected BSDE is a triplet $(Y^\Re,\widetilde{Y}^\Re,Z^\Re)$ satisfying

$$Y_T^\Re=\widetilde{Y}_T^\Re \ := \ g(X_T)\in\mathcal{O}_{X_T}$$

and, for $j\le\kappa-1$ and $t\in[r_j,r_{j+1})$,

$$\begin{cases}\widetilde{Y}_t^\Re &= Y_{r_{j+1}}^\Re+\int_t^{r_{j+1}} f(\Theta_u^\Re)\mathrm{d}u-\int_t^{r_{j+1}}(Z_u^\Re)'\mathrm{d}W_u\ ,\\ Y_t^\Re &= \mathcal{R}\left(t\ ,\ X_t\ ,\ \widetilde{Y}_t^\Re\right)\ ,\end{cases} \tag{2.4}$$

with $\Theta^\Re=(X,\widetilde{Y}^\Re,Z^\Re)$.
Here, $g:\mathbb{R}^d\mapsto\mathbb{R}^d$, $f:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{M}^d\mapsto\mathbb{R}^d$ are L-Lipschitz continuous and

$$\mathcal{R}^\ell(t,x,y):=y^\ell+([l^\ell(x)-y^\ell]^+-[y^\ell-h^\ell(x)]^+)\mathbf{1}_{\{t\in\Re\}}\ ,$$

for $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$, $\ell\in\{1,\dots,d\}$.

Observe that

$$Y_t^\Re \ = \ \widetilde{Y}_t^\Re\ ,\ \text{for } t\notin\Re\setminus\{T\}\ . \tag{2.5}$$

Remark 2.2. Under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, such a solution can be defined by backward induction. At each step the existence and uniqueness in $\mathcal{S}^2(\mathbb{R}^d)\times\mathcal{H}^2(\mathbb{M}^d)$ follow from e.g. [11]. Here, for $p\ge1$ and $E=\mathbb{M}^d$ or $E=\mathbb{M}^{d^2,d}$, $\mathcal{H}^p(E)$ is the set of progressively measurable E-valued processes V satisfying

$$\|V\|_{\mathcal{H}^p} \ := \ \|\left(\int_0^T |V_r|^2\mathrm{d}r\right)^{\frac12}\|_{L^p} \ < \ \infty\ .$$

Remark 2.3. The case where $(Y^{\Re}, X)$ takes values in $\mathbb{R}^n \times \mathbb{R}^d$ with $n \neq d$ can be treated in our framework. Indeed, if $d < n$, we can set $X^i := 0$, i.e. $b^i = 0$, $\sigma^{i\cdot} = 0$ and $X_0^i = 0$, for $i > d$. Recall that we make no ellipticity assumption. If $d > n$, we can set $g^i = f^i := 0$ which implies $Y^i = 0$, for $i > n$, and work with $\mathcal{O}_x \times [-\epsilon, \epsilon]^{d-n}$, $\epsilon > 0$, $x \in \mathbb{R}^d$.

We provide in the Appendix useful *a priori* estimates for "reflected" BSDEs in a somehow abstract setting. In our framework, they read as follows.

Proposition 2.1. *Under* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$*, the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re}|^2 + |Y_t^{\Re}|^2\Big] + ||Z^{\Re}||^2_{\mathcal{H}^2} \le C_L \kappa \,.$$

Moreover, if $(\mathbf{H}f)$*:* f^ℓ *depends on* $z^{\cdot i}$ *only for* $i = \ell$ *(i.e.* $\nabla_{z^{\cdot i}} f^\ell \mathbf{1}_{\{i \neq \ell\}} = 0$*, if* $f \in C^1$*), holds, then*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re}|^2 + |Y_t^{\Re}|^2\Big] \le C_L \,.$$

Proof. It suffices to apply Proposition 7.1 in the Appendix, with $\eta_r = |X_r|$ and $\xi_r = |h(X_r)| \vee |l(X_r)|$, $r \in \Re$, recall (2.2). $\square$

2.2 Dependence on the parameters

We now present some estimates on the variation in the solution of (2.4) induced by a variation in the data. Later on, this will allow us to work with smooth parameters (f, g, etc.) before turning to the general case by an approximation argument (see e.g. Proposition 4.2).

In the rest of this section, we consider two discretely reflected BSDEs constructed as follows.
For $i \in \{1, 2\}$, let X^i be an element of $\mathcal{S}^2(\mathbb{R}^d)$, f_i, g_i be L-Lipschitz continuous functions and h_i, l_i boundaries satisfying $(\mathbf{H}b1)$. We denote by $(Y^{\Re,i}, \widetilde{Y}^{\Re,i}, Z^{\Re,i})$ the solutions of the discretely reflected BSDE associated to these two sets of data and $\Theta^{\Re,i} := (X^i, \widetilde{Y}^{\Re,i}, Z^{\Re,i})$. We then define $\delta Y^{\Re} := Y^{\Re,1} - Y^{\Re,2}$, $\delta\widetilde{Y}^{\Re} := \widetilde{Y}^{\Re,1} - \widetilde{Y}^{\Re,2}$, $\delta Z^{\Re} := Z^{\Re,1} - Z^{\Re,2}$ and $\delta X := X^1 - X^2$, $\delta f := f_1(\Theta^{\Re,1}) - f_2(\Theta^{\Re,1})$, $\delta g := g_1(X^1) - g_2(X^1)$, $\delta h := h_1(X^1) - h_2(X^1)$ and $\delta l := l_1(X^1) - l_2(X^1)$.

Proposition 2.2. *Under* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$*, the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\delta Y_t^{\Re}|^2\Big] + ||\delta Z||^2_{\mathcal{H}^2} \le C_L \bigg(\kappa \mathbb{E}\Big[\max_{r\in\Re}(|\delta X_r|^2 + |\delta h_r|^2 + |\delta l_r|^2)\Big] + ||\delta f||^2_{\mathcal{H}^2} + \|\delta g_T\|^2_{L^2}\bigg) \,.$$

The proof of this result requires the following Lemma whose proof uses a key argument which will be very important below when studying the convergence of Euler scheme's type approximation of (2.4).

Lemma 2.1. *Let* $(\mathbf{H}x1)$*-*$(\mathbf{H}b1)$ *hold. Then for each* $r \in \Re \setminus \{T\}$ *and* $\ell \in \{1, \ldots, d\}$*, there exists* S_r^ℓ*,* Q_r^ℓ *in* $\mathcal{F}_r$ *such that* $S_r^\ell \cap Q_r^\ell = \emptyset$ *and*

$$\begin{aligned} |(Y_r^{\Re,1})^\ell - (Y_r^{\Re,2})^\ell| \quad &\leq \quad |(\widetilde{Y}_r^{\Re,1})^\ell - (\widetilde{Y}_r^{\Re,2})^\ell| \mathbf{1}_{S_r^\ell} \\ &\quad + \Big(|l_1^\ell(X_r^1) - l_2^\ell(X_r^2)| + |h_1^\ell(X_r^1) - h_2^\ell(X_r^2)| \Big) \mathbf{1}_{Q_r^\ell} \ . \end{aligned}$$

Proof. For ease of notations, we work with $d = 1$ and omit the exponent ℓ. Appropriate S_r and Q_r are constructed by considering different disjoint cases, depending on the position of $\widetilde{Y}_r^{\Re,1}$ and $\widetilde{Y}_r^{\Re,2}$.

1.a On $\{l_1(X_r^1) < \widetilde{Y}_r^{\Re,1} < h_1(X_r^1)\}$, three different cases may occur depending on the position of $\widetilde{Y}^{\Re,2}$.
(i) On $\{l_2(X_r^2) < \widetilde{Y}_r^{\Re,2} < h_2(X_r^2)\}$, we have $Y_r^{\Re,1} - Y_r^{\Re,2} = \widetilde{Y}_r^{\Re,1} - \widetilde{Y}_r^{\Re,2}$.
(ii) On $\{\widetilde{Y}_r^{\Re,2} \leq l_2(X_r^2)\}$, we have $Y_r^{\Re,2} = \mathcal{P}(X_r^2, \widetilde{Y}_r^{\Re,2}) = l_2(X_r^2)$. If $l_2(X_r^2) \leq \widetilde{Y}_r^{\Re,1}$, then $0 \leq Y_r^{\Re,1} - Y_r^{\Re,2} = \widetilde{Y}_r^{\Re,1} - l_2(X_r^2) \leq \widetilde{Y}_r^{\Re,1} - \widetilde{Y}_r^{\Re,2}$. If $l_2(X_r^2) > \widetilde{Y}_r^{\Re,1}$, then $0 \leq l_2(X_r^2) - \widetilde{Y}_r^{\Re,1} = Y_r^{\Re,2} - Y_r^{\Re,1} \leq l_2(X_r^2) - l_1(X_r^1)$.
(iii) On $\{h_2(X_r^2) \leq \widetilde{Y}_r^{\Re,2}\}$, similar arguments based on the comparison between $h_2(X_r^2)$ and $\widetilde{Y}_r^{\Re,1}$ lead to $|Y_r^{\Re,1} - Y_r^{\Re,2}| \leq |\widetilde{Y}_r^{\Re,1} - \widetilde{Y}_r^{\Re,2}|$ on $\{\widetilde{Y}_r^{\Re,1} \leq h_2(X_r^2)\}$ and $|Y_r^{\Re,1} - Y_r^{\Re,2}| \leq |h_2(X_r^2) - h_1(X_r^1)|$ on $\{h_2(X_r^2) < \widetilde{Y}_r^{\Re,1}\}$.

1.b We now study the case $\{\widetilde{Y}_r^{\Re,1} \leq l_1(X_r^1)\}$ which implies $Y_r^{\Re,1} = l_1(X_r^1)$.
(i) On $\{\widetilde{Y}_r^{\Re,2} \leq l_2(X_r^2)\}$, we have $Y_r^{\Re,1} - Y_r^{\Re,2} = l_1(X_r^1) - l_2(X_r^2)$.
(ii) On $\{l_2(X_r^2) < \widetilde{Y}_r^{\Re,2} < h_2(X_r^2)\}$, there are two disjoint cases. On $\{Y_r^{\Re,2} < Y_r^{\Re,1}\}$, $0 \leq Y_r^{\Re,1} - Y_r^{\Re,2} \leq l_1(X_r^1) - l_2(X_r^2)$. On $\{Y_r^{\Re,2} \geq Y_r^{\Re,1}\}$, $0 \leq Y_r^{\Re,2} - Y_r^{\Re,1} \leq \widetilde{Y}_r^{\Re,2} - \widetilde{Y}_r^{\Re,1}$.
(iii) Finally on $\{\widetilde{Y}_r^{\Re,2} \geq h_2(X_r^2)\}$, we also have two disjoint cases. On $\{h_2(X_r^2) > Y_r^{\Re,1}\}$, $0 \leq Y_r^{\Re,2} - Y_r^{\Re,1} \leq \widetilde{Y}_r^{\Re,2} - \widetilde{Y}_r^{\Re,1}$. On $\{h_2(X_r^2) \leq Y_r^{\Re,1}\}$, $0 \leq Y_r^{\Re,1} - h_2(X_r^2) \leq h_1(X_r^1) - h_2(X_r^2)$.

1.c By symmetry, the case $\widetilde{Y}_r^{\Re,1} \geq h_1(X_r^1)$ is handled similarly. □

We now provide the proof of Proposition 2.2.

Proof of Proposition 2.2. The proof of this Proposition relies on the abstract results of Proposition 7.1 in the Appendix. For $t \in [r_j, r_{j+1})$, we have

$$\delta \widetilde{Y}_t^\Re = \delta Y_{r_{j+1}}^\Re + \int_t^{r_{j+1}} \hat{f}(u) \mathrm{d}u - \int_t^{r_{j+1}} (\delta Z_u^\Re)' \mathrm{d}W_u \ ,$$

where $\hat{f} := \delta f + f_2(\Theta^{\Re,1}) - f_2(\Theta^{\Re,2})$.
Since f_2 is L-Lipschitz continuous, we have

$$|\hat{f}_u|^2 \leq C_L(|\eta_u|^2 + |\delta \widetilde{Y}_u^\Re|^2 + |\delta Z_u^\Re|^2) \ , \text{ with } \eta_u := |\delta f_u| + |\delta X| \ .$$

Moreover, using Lemma 2.1, we can set $\xi := 2L|\delta X| + |\delta l| + |\delta h|$, since h_2 and l_2 are L-Lipschitz continuous.

The proof is then concluded by appealing to Proposition 7.1 and observing that $|\delta Y_T^{\Re}| \le L|\delta X_T| + |\delta g_T|$, since g_2 is L-Lipschitz continuous. □

3 Representation results for $Z^{\Re}$

In this section, we provide different representations for $Z^{\Re}$. The first two ones are stated in terms of the Malliavin derivatives of $(X, Y^{\Re}, Z^{\Re})$, the last one is based on their associated "first variation" processes.
In order ensure that $(X, Y^{\Re}, Z^{\Re})$ are "smooth" enough, we shall work under the additional assumption:

- $(\mathbf{H}r)$: h, l, f, b and σ are C_b^1.

These representations will allow us to provide regularity results for $(Y^{\Re}, Z^{\Re})$ under $(\mathbf{H}r)$. This assumption will then be relieved by using an approximation argument based on Proposition 2.2 above.

3.1 Malliavin differentiability of $(X, Y^{\Re}, Z^{\Re})$

In the sequel, we denote by $\mathbb{D}^{1,2}$ the space of random variable F which are differentiable in the Malliavin sense and such that

$$\|F\|_{L^2}^2 + \int_0^T \|D_t F\|_{L^2}^2 \mathrm{d}t < \infty \, .$$

Here, $D_t F$ denotes the Malliavin derivative of F at time $t \le T$, see e.g. [16].
We also consider the space $\mathbb{L}^{1,2}$ of adapted processes V such that, after possibly passing to a suitable version, $V_s \in \mathbb{D}^{1,2}$ for all $s \le T$ and

$$||V||_{\mathcal{H}^2} + \int_0^T ||D_t V||_{\mathcal{H}^2} \mathrm{d}t < \infty \, .$$

In the following, we shall always work with a suitable version if necessary.

Under $(\mathbf{H}r)$, X belongs to $\mathbb{L}^{1,2}$, see [16]. It follows that $\mathcal{R}^\ell(r, X, F) \in \mathbb{D}^{1,2}$ whenever $F \in \mathbb{D}^{1,2}$ and

$$D_t \mathcal{R}^\ell(r, X, F) = D_t F^\ell + (D_t l^\ell(X_r) - D_t F^\ell) \mathbf{1}_{\{l^\ell(X_r) > F^\ell\}} - (D_t F^\ell - D_t h^\ell(X_r)) \mathbf{1}_{\{h^\ell(X_r) < F^\ell\}} . \tag{3.1}$$

Indeed, by a direct adaptation of the proof of Proposition 1.2.3 in [16] we deduce that, for $G \in \mathbb{D}^{1,2}$, $[G]^+$ belongs to $\mathbb{D}^{1,2}$ and $D_t[G]^+ = \alpha(D_t G)$ where α is a random variable bounded by 1 satisfying $\mathbf{1}_{\{G>0\}}\alpha = \mathbf{1}_{\{G>0\}}$. Thus Proposition 1.3.7 in [16] implies that $D_t[G]^+ = D_t G \mathbf{1}_{\{G>0\}}$, if $G \in \mathbb{D}^{1,2}$.

Combining (2.4), (3.1), and Proposition 5.3 in [11] with an induction argument, we obtain that $(\widetilde{Y}^{\Re}, Z^{\Re})$ belongs to $\mathbb{L}^{1,2}$ and that a version of $D_t((\widetilde{Y}^{\Re})^\ell, (Z^{\Re})^{\cdot\ell})$ is given by the solution in $\mathcal{S}^2(\mathbb{R}^d) \times \mathcal{H}^2(\mathbb{M}^d)$ of

$$\begin{aligned} D_t(\widetilde{Y}^{\Re}_s)^\ell &= D_t(Y^{\Re}_{r_{j+1}})^\ell + \int_s^{r_{j+1}} (\nabla_x f^\ell(\Theta^{\Re}_u) D_t X_u + \nabla_y f^\ell(\Theta^{\Re}_u) D_t \widetilde{Y}^{\Re}_u) \mathrm{d}u \qquad (3.2) \\ &\quad + \int_s^{r_{j+1}} \sum_{i=1}^d \nabla_{z^{\cdot i}} f^\ell(\Theta^{\Re}_u) D_t (Z^{\Re}_u)^{\cdot i} \mathrm{d}u - \int_s^{r_{j+1}} \sum_{k=1}^d D_t (Z^{\Re}_u)^{k\ell} \mathrm{d}W^k_u \,, \end{aligned}$$

for $s \in [r_j, r_{j+1})$, $j < \kappa$, with the terminal condition

$$D_t(\widetilde{Y}^{\Re}_T)^\ell = \nabla g^\ell(X_T) D_t X_T \,.$$

We conclude this section with some *a priori* estimates that will be used later on. The first one concerning DX is standard, we therefore omit the proof (see e.g. [16]).

Proposition 3.1. *Let* $(\mathbf{H}r)$ *hold. Then, for all* $p \geq 2$

$$\sup_{s \leq u \wedge t} \|D_s X_t - D_s X_u\|_{L^p} + \|(D_t X - D_u X)\mathbf{1}_{[t\vee u, T]}\|_{\mathcal{S}^p} \leq C^p_L \, |t-u|^{\frac{1}{2}} \,, \; t, u \leq T. \quad (3.3)$$

and

$$\|\sup_{s \leq T} |D_s X| \,\|_{\mathcal{S}^p} \;\; \leq \;\; C^p_L \,. \quad (3.4)$$

We now turn to the study of $(DY^{\Re}, DZ^{\Re})$. For ease of notations, we will from now on denote by β a $\mathcal{F}_T$-measurable positive random variable, whose value may change from line to line, but satisfies

$$\mathbb{E}[\beta^p] \;\; \leq \;\; C^p_L \,, \; \forall p \geq 1 \,.$$

Proposition 3.2. *Let* $(\mathbf{H}r)$ *hold. Then, for* $s \leq t \leq T$*,*

$$|D_s \widetilde{Y}^{\Re}_t|^2 + |D_s Y^{\Re}_t|^2 + \mathbb{E}_t\left[\int_t^T \sum_{\ell=1}^d |D_s (Z^{\Re}_u)^{\cdot\ell}|^2 \mathrm{d}u\right] \;\; \leq \;\; \kappa \mathbb{E}_t[\beta] \;. \quad (3.5)$$

If $(\mathbf{H}f)$ *holds, then, for* $p \geq 2$*,*

$$|D_s \widetilde{Y}^{\Re}_t|^p + |D_s Y^{\Re}_t|^p + \mathbb{E}_t\left[\sum_{\ell=1}^d \int_t^{\tau^\ell_j} |D_s (Z^{\Re}_u)^{\cdot\ell}|^2 \mathrm{d}u\right] \;\; \leq \;\; \mathbb{E}_t[\beta] \;, \quad (3.6)$$

and

$$|D_s \widetilde{Y}^{\Re}_t|^p + |D_s Y^{\Re}_t|^p \;\; \leq \;\; \mathbb{E}_t[\beta] \;, \quad (3.7)$$

for $j \leq \kappa - 1$, $t \in [r_j, r_{j+1})$, $s \leq t$, *where*

$$\tau^\ell_j = \inf\{t \in \Re \mid t \geq r_{j+1}, \; (\widetilde{Y}^{\Re}_t)^\ell \notin [l^\ell(X_t), h^\ell(X_t)]\} \wedge T \,, \; j \leq \kappa - 1 \,, \; \ell \leq d \,. \quad (3.8)$$

Proof. Recall that for $F \in \mathbb{D}^{1,2}$, $DF = (D^1F, \ldots, D^dF)$ where D^i denotes the Malliavin derivatives with respect to W^i. Fix $q \in \{1, \ldots, d\}$, by (3.2), we have for all $t \le s \in [r_j, r_{j+1})$ and $j < \kappa$

$$\begin{aligned} D^q_t \widetilde{Y}^{\Re}_s &= D^q_t Y^{\Re}_{r_{j+1}} - \int_s^{r_{j+1}} (D^q_t Z^{\Re}_u)' \mathrm{d}W_u \\ &\quad + \int_s^{r_{j+1}} \Big(\nabla_x f(\Theta^{\Re}_u) D^q_t X_u + \nabla_y f(\Theta^{\Re}_u) D^q_t \widetilde{Y}^{\Re}_u + \nabla_z f(\Theta^{\Re}_u) D^q_t Z^{\Re}_u \Big) \, \mathrm{d}u \, . \end{aligned}$$

Since f is C^1_b under $(\mathbf{H}r)$, (7.2) of the Appendix holds with $\eta = |D^q_t X|$. Clearly $(\mathbf{A}f)$ holds under $(\mathbf{H}f)$.
Moreover, it follows from (3.1), that $(D^q_t Y^{\Re}, D^q_t \widetilde{Y}^{\Re})$ satisfies $(\mathbf{A}0)$ (take $S^\ell_r = \{(Y^{\Re}_r)^\ell \in [l^\ell(X_r), h^\ell(X_r)]\}$, for $r \in \Re$ and $\ell \in \{1 \ldots d\}$).
The result is then a direct application of Proposition 7.1 and Corollary 7.1. $\square$

Similar arguments based on Proposition 7.1 also lead to

Proposition 3.3. *Under* $(\mathbf{H}r)$, *we have, for all* $t \le T$ *and* $r, s \le t$,

$$\begin{aligned} |D_s \widetilde{Y}^{\Re}_t - D_r \widetilde{Y}^{\Re}_t|^2 &+ |D_s Y^{\Re}_t - D_r Y^{\Re}_t|^2 \\ &+ \mathbb{E}_t\Bigg[\int_t^T \sum_{\ell=1}^d |D_s (Z^{\Re}_u)^{\cdot\ell} - D_r (Z^{\Re}_u)^{\cdot\ell}|^2 \mathrm{d}u \Bigg] \le \kappa \mathbb{E}_t[\beta] \, |s - r| \, . \end{aligned}$$

Under $(\mathbf{H}f)$,

$$\begin{aligned} |D_s \widetilde{Y}^{\Re}_t - D_r \widetilde{Y}^{\Re}_t|^2 &+ |D_s Y_t - D_r Y_t|^2 \\ &+ \mathbb{E}_t\Bigg[\sum_{\ell=1}^d \int_t^{\tau^\ell_j} |D_s (Z^{\Re}_u)^{\cdot\ell} - D_r (Z^{\Re}_u)^{\cdot\ell}|^2 \mathrm{d}u \Bigg] \le \mathbb{E}_t[\beta] \, |s - r| \, , \end{aligned}$$

for $j \le \kappa - 1$, $t \in [r_j, r_{j+1})$ *and* $r, s \le t$.

3.2 Representation in terms of Malliavin derivatives of $(X, Y^{\Re}, Z^{\Re})$

It follows from [16] and (2.4), viewed in a forward way, that $(D_t Y^{\Re}_t)_{t \le T}$ is a version of $Z^{\Re}$. Hence, (3.2) implies that $Z^{\Re}$ admits a version satisfying

$$(Z^{\Re}_t)' = \mathbb{E}_t\Bigg[D_t Y^{\Re}_{r_{j+1}} + \int_t^{r_{j+1}} (\nabla_x f(\Theta^{\Re}_u) D_t X_u + \nabla_y f(\Theta^{\Re}_u) D_t \widetilde{Y}^{\Re}_u + \sum_{i=1}^d \nabla_{z^{\cdot i}} f(\Theta^{\Re}_u) D_t (Z^{\Re}_u)^{\cdot i}) \mathrm{d}u \Bigg] \tag{3.9}$$

for each $j \le \kappa - 1$ and $t \in [r_j, r_{j+1})$.

Following the arguments of [3], we can get rid of the term $D_t Y^{\Re}_{r_{j+1}}$ in the above expression.

Corollary 3.1. *Let* ($\mathbf{H}r$) *hold. Then, for each* $\ell \in \{1,\ldots,d\}$, *there is a version of* $(Z^{\Re})^{\cdot\ell}$ *such that, for each* $j \le \kappa-1$ *and* $s \le t \in [r_j, r_{j+1})$,

$$((Z^{\Re}_t)^{\cdot\ell})' = \mathbb{E}_t\Big[\nabla\phi^\ell_{\tau^\ell_j} D_t X_{\tau^\ell_j} \;+\; \int_t^{\tau^\ell_j} \nabla_x f^\ell(\Theta^{\Re}_u) D_t X_u \mathrm{d}u \tag{3.10}$$
$$+ \int_t^{\tau^\ell_j}\left(\nabla_y f^\ell(\Theta^{\Re}_u) D_t \widetilde{Y}^{\Re}_u + \sum_{i=1}^d \nabla_{z^{\cdot i}} f^\ell(\Theta^{\Re}_u) D_t (Z^{\Re}_u)^{\cdot i}\right)\mathrm{d}u\Bigg],$$

where, for $r \in \Re$,

$$\nabla\phi^\ell_r := \nabla g^\ell(X_r)\mathbf{1}_{\{r=T\}} + \Big(\nabla l^\ell(X_r)\mathbf{1}_{\{l^\ell(X_r)>(\widetilde{Y}^{\Re}_r)^\ell\}} + \nabla h^\ell(X_r)\mathbf{1}_{\{h^\ell(X_r)<(\widetilde{Y}^{\Re}_r)^\ell\}}\Big)\mathbf{1}_{\{r<T\}}\ .$$

Proof. For $\ell \in \{1,\ldots,d\}$, $q \le \kappa-1$, we denote by ξ^ℓ_q the random index such that $r_{\xi^\ell_q} = \tau^\ell_q$ (recall the definition of τ^ℓ_q in Proposition 3.2). On $\{\tau^\ell_q = r_{q+1}\}$, the result is obvious. On $\{\tau^\ell_q > r_{q+1}\}$, summing up from q to ξ^ℓ_q in (3.2) applied to $s = r_{q+1}$ and using (3.1) leads to

$$\begin{aligned} D_t(\widetilde{Y}^{\Re}_s)^\ell &= \nabla\phi^\ell_{\tau^\ell_j} D_t X_{\tau^\ell_q} + \int_s^{\tau^\ell_q} (\nabla_x f^\ell(\Theta^{\Re}_u) D_t \chi_u + \nabla_y f^\ell(\Theta^{\Re}_u) D_t \widetilde{Y}^{\Re}_u)\mathrm{d}u \\ &\quad + \int_s^{\tau^\ell_q} \sum_{i=1}^d \nabla_{z^{\cdot i}} f^\ell(\Theta^{\Re}_u) D_t (Z^{\Re}_u)^{\cdot i}\mathrm{d}u - \int_s^{\tau^\ell_q} \sum_{k=1}^d D_t (Z^{\Re}_u)^{k\ell}\mathrm{d}W^k_u\ , \end{aligned} \tag{3.11}$$

for $t \le s \in [r_q, r_{q+1})$.
Since $(D_t(\widetilde{Y}^{\Re}_t)^\ell)_{t\le T}$ is a version of $((Z^{\Re}_t)^{\cdot\ell})_{t\le T}$, the required result is obtained by taking the conditional expectation in the above expression. □

Under ($\mathbf{H}f$), we can also get rid of the term $D_t Z^{\Re}$ in the expressions (3.9) and (3.10) by arguing as in [17] and [3]. Indeed, applying Itô's Lemma to $Y^{\Re}\Lambda^\ell$ with

$$\Lambda^\ell_t \;:=\; \exp\left\{\int_0^t \nabla_{z^{\cdot\ell}} f^\ell(\Theta^{\Re}_u)'\mathrm{d}W_u - \int_0^t \frac{1}{2}|\nabla_{z^{\cdot\ell}} f^\ell(\Theta^{\Re}_u)|^2\mathrm{d}u\right\}\ ,\ t \le T\ ,$$

we directly deduce from (3.11) the following alternative representation.

Corollary 3.2. *Let* ($\mathbf{H}r$) *and* ($\mathbf{H}f$) *hold. Then, there is a version of* $(Z^{\Re})^{\cdot\ell}$ *such that*

$$((Z^{\Re}_t)^{\cdot\ell})' = (\Lambda^\ell_t)^{-1}\mathbb{E}_t\left[\nabla\phi^\ell_{\tau^\ell_j}(\Lambda^\ell D_t X)_{\tau^\ell_j} + \int_t^{\tau^\ell_j}\Big(\nabla_x f^\ell(\Theta^{\Re}_u)(\Lambda^\ell D_t X)_u + \nabla_y f^\ell(\Theta^{\Re}_u)(\Lambda^\ell D_t \widetilde{Y}^{\Re})_u\Big)\mathrm{d}u\right]$$

for $s \le t \in [r_j, r_{j+1})$, $j \le \kappa-1$ *and* $\ell \in \{1,\ldots,d\}$.

Observe that this simplification is no more possible if f^ℓ depends of more than one columns of $Z^{\Re}$.

Remark 3.1. For later use, observe that:

$$\|\sup_{s\le t\le T}\Lambda^\ell_t\|_{L^p} \;\le\; C^p_L\ , \tag{3.12}$$

$$\|\sup_{u\le t\wedge s}|\Lambda^\ell_t(\Lambda^\ell_u)^{-1} - \Lambda^\ell_s(\Lambda^\ell_u)^{-1}|\,\|_{L^p} \;\le\; C^p_L|t-s|^{\frac{1}{2}}\ ,\ t,s \le T\ . \tag{3.13}$$

3.3 First variation processes associated to $(X, Y^{\Re}, Z^{\Re})$

In Section 3.4 below, we provide a representation of $Z^{\Re}$ in term of the first variation process of $(X, \widetilde{Y}^{\Re})$.
Under $(\mathbf{H}r)$, the first variation process ∇X of X is well defined and solves on $[0, T]$

$$\nabla X_t = I_d + \int_0^t \nabla_x b(X_r)\nabla X_r \mathrm{d}r + \int_0^t \sum_{j=1}^d \nabla_x \sigma^{\cdot j}(X_r)\nabla X_r \mathrm{d}W_r^j$$

where I_d is the identity matrix of $\mathbb{M}^d$. Its inverse $(\nabla X)^{-1}$ is the solution on $[0, T]$ of

$$\begin{aligned}(\nabla X)_t^{-1} &= I_d - \int_0^t (\nabla X)_r^{-1}\left[\nabla_x b(X_r) - \sum_{j=1}^d \nabla_x \sigma^{\cdot j}(X_r)\nabla\sigma^{\cdot j}(X_r)\right]\mathrm{d}r \\ &\quad - \int_0^t \sum_{j=1}^d (\nabla X)_r^{-1}\nabla_x\sigma^{\cdot j}(X_r)\mathrm{d}W_r^j .\end{aligned}$$

Recall the well-known relation between ∇X and DX:

$$D_t X_s = \nabla X_s (\nabla X_t)^{-1}\sigma(X_t)\mathbf{1}_{t\le s} \quad \text{for all } t, s \le T . \tag{3.14}$$

Remark 3.2. The following standard estimates hold:

$$||\nabla X||_{\mathcal{S}^p} + ||(\nabla X)^{-1}||_{\mathcal{S}^p} \le C_L^p . \tag{3.15}$$

Let us now consider the processes $(\nabla Y^{\Re}, \nabla\widetilde{Y}^{\Re}) \in \mathcal{S}^2(\mathbb{M}^d) \times \mathcal{S}^2(\mathbb{M}^d)$ and $\nabla Z^{\Re,i} \in \mathcal{H}^2(\mathbb{M}^d)$, $i \in \{1, \dots, d\}$, defined as the solutions of the coupled linear discretely "reflected" BSDEs:

$$\nabla Y_T^{\Re} = \nabla\widetilde{Y}_T^{\Re} := \nabla g(X_T)\nabla X_T$$

and, for $j \le \kappa - 1$, $t \in [r_j, r_{j+1})$, $\ell \in \{1, \dots, d\}$,

$$\begin{aligned}(\nabla\widetilde{Y}^{\Re})_t^{\ell\cdot} &= (\nabla Y^{\Re})_{r_{j+1}}^{\ell\cdot} + \int_t^{r_{j+1}} (\nabla_x f(\Theta_u^{\Re})\nabla X_u + \nabla_y f(\Theta_u^{\Re})\nabla\widetilde{Y}_u^{\Re} + \sum_{i=1}^d \nabla_{z^{\cdot i}} f^{\ell}(\Theta_u^{\Re})\nabla Z_u^{\Re,i})\mathrm{d}u \\ &\quad - \int_t^{r_{j+1}} \sum_{k=1}^d (\nabla Z_u^{\Re,\ell})^{k\cdot}\mathrm{d}W_u^k \end{aligned} \tag{3.16}$$

where $(\nabla Y^{\Re})^{\ell\cdot}$ is defined through the "pseudo-reflection"

$$\begin{aligned}(\nabla Y^{\Re})_t^{\ell\cdot} := (\nabla\widetilde{Y}^{\Re})_t^{\ell\cdot} &+ \Big((\nabla l^{\ell}(X_t)\nabla X_t - (\nabla\widetilde{Y}^{\Re})_t^{\ell\cdot})\mathbf{1}_{\{l^{\ell}(X_t)\ge(\widetilde{Y}_t^{\Re})^{\ell}\}} \\ &- ((\nabla\widetilde{Y}^{\Re})_t^{\ell\cdot} - \nabla h^{\ell}(X_t)\nabla X_t)\mathbf{1}_{\{h^{\ell}(X_t)\le(\widetilde{Y}_t^{\Re})^{\ell}\}}\Big)\mathbf{1}_{\{t\in\Re\}}.\end{aligned} \tag{3.17}$$

Observe that the system of coupled BSDEs (3.16) can be rewritten as:

$$\widetilde{U}^{\Re}_t = U^{\Re}_{r_{j+1}} + \int_t^{r_{j+1}} F(\nabla X_u, \widetilde{U}^{\Re}_u, V^{\Re}_u)\mathrm{d}u - \int_t^{r_{j+1}} V^{\Re}_u \mathrm{d}W_u \ , \ t \in [r_j, r_{j+1}) \ , \quad (3.18)$$

where F is a linear operator with random coefficient and values in $\mathbb{R}^{d^2}$, $(U^{\Re}, \widetilde{U}^{\Re}, V^{\Re})$ takes value in $\mathbb{R}^{d^2} \times \mathbb{R}^{d^2} \times \mathbb{M}^{d^2,d}$ and

$$\begin{aligned}
(U^{\Re})' &= [(\nabla Y^{\Re})^{1\cdot}, \dots, (\nabla Y^{\Re})^{d\cdot}], \\
(\widetilde{U}^{\Re})' &= [(\nabla \widetilde{Y}^{\Re})^{1\cdot}, \dots, (\nabla \widetilde{Y}^{\Re})^{d\cdot}], \\
(V^{\Re})' &= [\nabla Z^{\Re,1}, \dots, \nabla Z^{\Re,d}].
\end{aligned}$$

Thus, existence and uniqueness in $\mathcal{S}^2(\mathbb{R}^{d^2}) \times \mathcal{S}^2(\mathbb{R}^{d^2}) \times \mathcal{H}^2(\mathbb{M}^{d^2,d})$ follows easily from a simple induction argument.

Remark 3.3. Using (3.2) and (3.14), we observe that $(D_t Y^{\Re}, D_t \widetilde{Y}^{\Re}, (D_t (Z^{\Re})^{\cdot \ell})_{\ell \in \{1,\dots,d\}})$ and $(\nabla Y^{\Re} \nabla X_t^{-1} \sigma(X_t), \nabla \widetilde{Y}^{\Re} \nabla X_t^{-1} \sigma(X_t), (\nabla Z^{\Re,\ell} \nabla X_t^{-1} \sigma(X_t))_{\ell \in \{1,\dots,d\}})$ verify the same equation of type (3.18). By uniqueness of the solution, this implies that

$$\begin{aligned}
D_t(\widetilde{Y}^{\Re}_s)^{\ell} &= (\nabla \widetilde{Y}^{\Re})^{\ell \cdot}_s \nabla X_t^{-1} \sigma(X_t), \\
D_t(Y^{\Re}_s)^{\ell} &= (\nabla Y^{\Re})^{\ell \cdot}_s \nabla X_t^{-1} \sigma(X_t), \\
D_t(Z^{\Re}_s)^{\cdot \ell} &= \nabla Z^{\Re,\ell}_s \nabla X_t^{-1} \sigma(X_t),
\end{aligned}$$

for $\ell \in \{1, \dots, d\}$, $t \le s \le T$.

Remark 3.4. By using the same arguments as in proof of Proposition 3.2, we easily deduce that, under $(\mathbf{H}r)$ and $(\mathbf{H}f)$,

$$|\nabla \widetilde{Y}^{\Re}_t|^p + |\nabla Y^{\Re}_t|^p \le \mathbb{E}_t[\beta^p] \ , \quad (3.19)$$

for $t \le T$, $p \ge 2$. Recall that β denotes a $\mathcal{F}_T$-measurable positive random variable, whose value may change from line to line, but satisfies $\mathbb{E}[\beta^p] \le C_L^p$ for all $p \ge 1$.

3.4 Representation in terms of $(\nabla X, \nabla \widetilde{Y}^{\Re})$

Combining Corollary 3.2, (3.14) and Remark 3.3, we deduce this last representation for $(Z^{\Re})^{\cdot \ell}$.

Corollary 3.3. *Let* $(\mathbf{H}r)$ *and* $(\mathbf{H}f)$ *hold. Then, for each* $\ell \in \{1, \dots, d\}$*, there is a version of* $(Z^{\Re})^{\cdot \ell}$ *such that*

$$\begin{aligned}
((Z^{\Re}_t)^{\cdot \ell})' &= (\Lambda^{\ell}_t)^{-1} \mathbb{E}_t\Big[\nabla \phi^{\ell}_{\tau_j} (\Lambda^{\ell} \nabla X)_{\tau_j} \\
&+ \int_t^{\tau_j} \Big(\nabla_x f^{\ell}(\Theta^{\Re}_u)(\Lambda^{\ell} \nabla X)_u + \nabla_y f^{\ell}(\Theta^{\Re}_u)(\Lambda^{\ell} \nabla \widetilde{Y}^{\Re})_u \Big) \mathrm{d}u \Big] \nabla X_t^{-1} \sigma(X_t),
\end{aligned}$$

for $s \le t \in [r_j, r_{j+1})$*,* $j \le \kappa - 1$.

4 Regularity results

Based on the representations of the previous section and the stability result of Proposition 2.2, we can now provide one of the main results of this paper which concerns the regularity of $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re})$. Namely, we study the quantities

$$||\widetilde{Y}^{\Re} - \mathcal{D}^{\pi}\widetilde{Y}^{\Re}||_{\mathcal{H}^2} \quad \text{and} \quad ||Z^{\Re} - P^{\pi}Z^{\Re}||_{\mathcal{H}^2} , \tag{4.1}$$

where $\pi = \{0 =: t_0 < t_1 < \ldots < t_n := T\}$ is a partition of the time interval $[0, T]$ with modulus $|\pi|$ and such that $\Re \subset \pi$, $\mathcal{D}^{\pi}$ is the usual piecewise approximation operator defined on $\mathcal{H}^2(\mathbb{R}^d)$ by

$$\mathcal{D}^{\pi}V \;\; := \;\; \sum_{i=0}^{n-1} V_{t_i}\mathbf{1}_{[t_i,t_{i+1})} \; + \; V_T\mathbf{1}_{\{T\}} \; ,$$

and P^{π} is defined on $\mathcal{H}^2(\mathbb{M}^d)$ by

$$P^{\pi}V := \sum_{i=0}^{n-1} \bar{V}^{\pi}_{t_i}\mathbf{1}_{[t_i,t_{i+1})} \;\; \text{with} \;\; \bar{V}^{\pi}_{t_i} := \frac{1}{t_{i+1} - t_i}\mathbb{E}\left[\int_{t_i}^{t_{i+1}} V_s \mathrm{d}s \mid \mathcal{F}_{t_i}\right] . \tag{4.2}$$

Remark 4.1. $P^{\pi}V$ is the best $L^2(\Omega \times [0, T])$-approximation of V by adapted processes which are constant on each interval $[t_i, t_{i+1})$.

As shown in [4], [3], [5] and [15], the control of such quantities plays a central role in the study of Euler scheme's type approximations of BSDEs and it will be used in the next sections.

4.1 Regularity of $Y^{\Re}$

Proposition 4.1. *Set $\alpha(\kappa) = \kappa$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$ and $\alpha(\kappa) = 1$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$, then the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_t - (\mathcal{D}^{\pi}\widetilde{Y}^{\Re})_t|^2\Big] \le C_L\alpha(\kappa)|\pi| \; .$$

Proof. Noting that, for $j < \kappa$ and $t \in [t_i, t_{i+1}) \subset [r_j, r_{j+1}]$,

$$|\widetilde{Y}^{\Re}_t - \widetilde{Y}^{\Re}_{t_i}|^2 \;\; \le \;\; 2\left(\int_{t_i}^{t_{i+1}} |f(X_u, \widetilde{Y}^{\Re}_u, Z^{\Re}_u)|^2\mathrm{d}u + \sup_{t\in[t_i,t_{i+1}]} |\int_t^{t_{i+1}} (Z^{\Re}_u)'\mathrm{d}W_u|^2\right) \; ,$$

it follows directly from Proposition 2.1, Proposition 4.2 below and Burkholder-Davis-Gundy inequality, that

$$\mathbb{E}\Big[|\widetilde{Y}^{\Re}_t - \widetilde{Y}^{\Re}_{t_i}|^2\Big] \le C_L\alpha(\kappa)|\pi| \; ,$$

which concludes the proof. □

The following immediate Corollary provides an estimate of the first term of (4.1).

Corollary 4.1. *Set $\alpha(\kappa)=\kappa$ under* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$ *and $\alpha(\kappa)=1$ under* $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$*, then the following holds*

$$||\widetilde{Y}^{\Re}-\mathcal{D}^{\pi}\widetilde{Y}^{\Re}||^2_{\mathcal{H}^2}\leq C_L\alpha(\kappa)|\pi|\ .$$

We now state the Proposition which was used in the proof of Proposition 4.1. Observe that it provides a "weak" bound on $Z^{\Re}$.

Proposition 4.2. *Set $\alpha(\kappa)=\kappa$ under* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$ *and $\alpha(\kappa)=1$ under* $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$*. There is a version of $Z^{\Re}$ such that*

1. *For $s\leq t\leq T$, we have*

$$\mathbb{E}\Big[\int_s^t |Z^{\Re}_u|^2 \mathrm{d}u\Big] \quad\leq\quad C_L\alpha(\kappa)|t-s|\ .$$

2. *If* $(\mathbf{H}r)$ *holds, then there is a version of $Z^{\Re}$ such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Z^{\Re}_t|^2\Big] \quad\leq\quad C_L\alpha(\kappa)\ .$$

Proof. 1. Assume that $(\mathbf{H}r)$ holds. Since $(D_t\widetilde{Y}^{\Re}_t)_{t\leq T}$ is a version of $(Z^{\Re}_t)_{t\leq T}$, the second claim is a straightforward consequence of Proposition 3.2 and Burkholder-Davis-Gundy inequality. This implies the first one under $(\mathbf{H}r)$.
2. We now assume that only $(\mathbf{H}x1)$ holds for X i.e. b and σ are L-Lipschitz continuous and $(\mathbf{H}b1)$ for h and l. Recall that g, f are also L-Lipschitz continuous. Let $(f_n)_{n\geq 0}$ be the sequence of smooth functions defined by

$$f_n(x,y,z) \quad=\quad \int_{\mathbb{R}^{d(d+2)}}\phi_n(x-\xi,y-\upsilon,z-\zeta)f(\xi,\upsilon,\zeta)\mathrm{d}\xi\mathrm{d}\upsilon\mathrm{d}\zeta\ ,$$

with $\phi_n(x,y,z)=n^{d(d+2)}\phi(n(x,y,z))$ and ϕ a compactly supported smooth probability density function on $\mathbb{R}^{d(d+2)}$. Since f is L-Lipschitz continuous, we have

$$||f-f_n||_\infty \quad\leq\quad \frac{C_L}{n}\ .$$

Let g_n resp. σ_n and b_n be defined similarly with g resp. σ and b in place of f, so that

$$||g-g_n||_\infty+||b-b_n||_\infty+||\sigma-\sigma_n||_\infty \quad\leq\quad \frac{C_L}{n}\ .$$

Let X^n be the diffusion associated to b_n and σ_n, and $(Y^{\Re,n},Z^{\Re,n})$ be the solution of (2.4) associated to f_n, g_n and X^n. Since by step 1. and $(\mathbf{H}x1)$

$$\mathbb{E}\Big[\int_s^t |Z^{\Re,n}_u|^2\mathrm{d}u\Big] \quad\leq\quad C_L\alpha(\kappa)|t-s|\ ,$$

for all $s,t\leq T$ and $n\geq 0$, the required result follows from step 1 and Proposition 2.2. $\square$

4.2 Regularity of $Z^{\Re}$

The estimate for the second term of (4.1) is a bit more involved. We shall adapt the proof of Proposition 5.2 in [3] to our framework.

We first prove a result for the general case. The difficulty, which does not appear in [3], comes from the fact that $DZ^{\Re}$ is in the expression of $Z^{\Re}$ and can be eliminated only when $(\mathbf{H}f)$ holds. It is overcome using the *a priori* estimates of the previous section.

Proposition 4.3. *Set* $\alpha(\kappa) = \kappa$ *under* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$ *and* $\alpha(\kappa) = 1$ *under* $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$*, then the following holds*

$$||Z^{\Re} - P^{\pi} Z^{\Re}||^2_{\mathcal{H}^2} \quad \leq \quad C_L \alpha(\kappa)(\kappa|\pi| + |\pi|^{\frac{1}{2}}).$$

Proof. 1. First observe that a similar approximation argument as the one used in step 2. of the proof of Proposition 4.2 allows to reduce our study to the case where $(\mathbf{H}r)$ holds. We shall therefore assume from now on that $(\mathbf{H}r)$ holds.
Since, by Remark 4.1,

$$||Z^{\Re} - P^{\pi} Z^{\Re}||_{\mathcal{H}^2} \quad \leq \quad ||Z^{\Re} - \mathcal{D}^{\pi} Z^{\Re}||_{\mathcal{H}^2} \ ,$$

it suffices to show that the last term is bounded by $C_L \alpha(\kappa)(\kappa|\pi| + |\pi|^{\frac{1}{2}})$.
For each $\ell \in \{1, \ldots, d\}$ and $s \leq t \in [t_i, t_{i+1}) \subset [r_j, r_{j+1}]$, we define $V^{\ell,j}_{s,t}$ by

$$\mathbb{E}_t\left[\nabla\phi^{\ell}_{\tau^{\ell}_j}(D_s X)_{\tau^{\ell}_j} + \int_s^{\tau^{\ell}_j}\left(\nabla_x f^{\ell}(\Theta^{\Re}_u) D_s X_u + \nabla_y f^{\ell}(\Theta^{\Re}_u) D_s \widetilde{Y}^{\Re}_u + \sum_{k=1}^{d} \nabla_{z^{\cdot i}} f^{\ell}(\Theta^{\Re}_u) D_s (Z^{\Re}_u)^{\cdot k}\right) \mathrm{d}u\right].$$

After possibly passing to a suitable version of $Z^{\Re}$, we observe that

$$|(Z^{\Re}_t)^{\cdot \ell} - (Z^{\Re}_{t_i})^{\cdot \ell}| \leq |V^{\ell,j}_{t,t} - V^{\ell,j}_{t_i,t}| + |V^{\ell,j}_{t_i,t} - V^{\ell,j}_{t_i,t_i}| \ , \tag{4.3}$$

recall Corollary 3.1. Defining i_j through $t_{i_j} = r_j$, $j \leq \kappa$, we shall prove the following controls

$$\sum_{j=0}^{\kappa-1} \sum_{k=i_j}^{i_{j+1}-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[|V^{\ell,j}_{t,t} - V^{\ell,j}_{t_k,t}|^2\Big] \, \mathrm{d}t \leq C_L \alpha(\kappa)|\pi| \tag{4.4}$$

and

$$\sum_{j=0}^{\kappa-1} \sum_{k=i_j}^{i_{j+1}-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[|V^{\ell,j}_{t_k,t} - V^{\ell,j}_{t_k,t_k}|^2\Big] \, \mathrm{d}t \leq C_L \alpha(\kappa)(\kappa|\pi| + |\pi|^{\frac{1}{2}}) \ . \tag{4.5}$$

2.a We first study (4.4). We have for $t \in [t_i, t_{i+1}) \subset [r_j, r_{j+1}]$

$$\begin{aligned}\left|V^{\ell,j}_{t,t} - V^{\ell,j}_{t_i,t}\right| \leq C_L \mathbb{E}_t\Bigg[&\int_t^{\tau^{\ell}_j} |D_t X_u - D_{t_i} X_u| + |D_t \widetilde{Y}^{\Re}_u - D_{t_i} \widetilde{Y}^{\Re}_u| + \sum_{k=1}^{d} |D_t (Z^{\Re}_u)^{\cdot k} - D_{t_i} (Z^{\Re}_u)^{\cdot k}| \mathrm{d}u \\ &+ \int_{t_i}^{t}\left(|D_{t_i} X_u| + |D_{t_i} \widetilde{Y}^{\Re}_u| + \sum_{k=1}^{d} |D_{t_i} (Z^{\Re}_u)^{\cdot k}|\right) \mathrm{d}u + \left|\nabla\phi^{\ell}_{\tau^{\ell}_j}(D_t X)_{\tau^{\ell}_j} - \nabla\phi^{\ell}_{\tau^{\ell}_j}(D_{t_i} X)_{\tau^{\ell}_j}\right|\Bigg]\end{aligned} \tag{4.6}$$

Observing that, by Cauchy-Schwartz inequality,

$$\mathbb{E}\Big[\Big|\int_{t_i}^{t}(|D_{t_i}X_u| \;+\; |D_{t_i}\widetilde{Y}^{\Re}_u| + \sum_{i=1}^{d}|D_{t_i}(Z^{\Re}_u)^{\cdot i}|)\mathrm{d}u\Big|^2\Big] \le$$

$$C_L|\pi|\mathbb{E}\Big[\int_{t_i}^{t}(|D_{t_i}X_u|^2 + |D_{t_i}\widetilde{Y}^{\Re}_u|^2 + \sum_{k=1}^{d}|D_{t_i}(Z^{\Re}_u)^{\cdot k}|^2)\mathrm{d}u\Big],$$

it follows from Proposition 3.2, Proposition 3.3, (3.3) and $(\mathbf{H}r)$ that

$$\mathbb{E}\Big[|V^{\ell,j}_{t,t} - V^{\ell,j}_{t_i,t}|^2\Big] \le C_L\alpha(\kappa)|\pi| \ . \tag{4.7}$$

2.b We now prove (4.5). Using the martingale property of $(V^{\ell,j}_{t_i,t})_{t\le T}$ on $[t_i,t_{i+1}]$, we obtain

$$\begin{aligned}\mathbb{E}\Big[|V^{\ell,j}_{t_i,t} - V^{\ell,j}_{t_i,t_i}|^2\Big] &\le \mathbb{E}\Big[|V^{\ell,j}_{t_i,t_{i+1}}|^2 - |V^{\ell,j}_{t_i,t_i}|^2\Big]\\ &\le \mathbb{E}\Big[|V^{\ell,j}_{t_{i+1},t_{i+1}}|^2 - |V^{\ell,j}_{t_i,t_i}|^2 + |V^{\ell,j}_{t_{i+1},t_{i+1}} - V^{\ell,j}_{t_i,t_{i+1}}|\,|V^{\ell,j}_{t_{i+1},t_{i+1}} + V^{\ell,j}_{t_i,t_{i+1}}|\Big],\end{aligned}$$

which by Proposition 3.3, $(\mathbf{H}r)$ and Cauchy-Schwartz inequality leads to

$$\mathbb{E}\Big[|V^{\ell,j}_{t_i,t} - V^{\ell,j}_{t_i,t_i}|^2\Big] \;\le\; \mathbb{E}\Big[|V^{\ell,j}_{t_{i+1},t_{i+1}}|^2 - |V^{\ell,j}_{t_i,t_i}|^2\Big] + C_L\alpha(\kappa)|\pi|^{\frac{1}{2}} \ . \tag{4.8}$$

To conclude the proof of (4.5), it remains to study the first term in the right-hand side of (4.8):

$$\begin{aligned}\Sigma^{\ell} \;&:=\; \sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\mathbb{E}\Big[|V^{\ell,j}_{t_{k+1},t_{k+1}}|^2 - |V^{\ell,j}_{t_k,t_k}|^2\Big]\\ &=\; \left(\mathbb{E}\Big[|V^{\ell,\kappa-1}_{T,T}|^2 - |V^{\ell,0}_{0,0}|^2\Big] + \sum_{j=1}^{\kappa-1}\mathbb{E}\Big[|V^{\ell,j-1}_{r_j,r_j}|^2 - |V^{\ell,j}_{r_j,r_j}|^2\Big]\right),\end{aligned}$$

so that, by Proposition 3.2,

$$\mathbb{E}\Big[\Sigma^{\ell}\Big] \;\le\; C_L\kappa\alpha(\kappa) \ .$$

This implies (4.5). □

As in the simply reflected case studied by [3], the estimate of Proposition 4.3 can be improved if we impose more regularity on the forward process and the boundaries. The main new difficulty due to our multidimensional setting is that the projection of $(Y^{\Re})^{\ell}$ is not well known: it could be equal to the upper or the lower boundary. This is overcome by appealing to the following Lemma which is proved at the end of this section.

Lemma 4.1. *Recall the definitions of $\nabla\phi^\ell$ and τ_j^ℓ in Proposition 3.2. Under* $(\mathbf{H}f)$*-*$(\mathbf{H}x2)$*-*$(\mathbf{H}b3)$*, the following holds*

$$|\mathbb{E}_{r_j}\Big[\nabla\phi^\ell_{\tau^\ell_{j-1}}\Lambda^\ell_{\tau^\ell_{j-1}}\nabla X_{\tau^\ell_{j-1}} - \nabla\phi^\ell_{\tau^\ell_j}\Lambda^\ell_{\tau^\ell_j}\nabla X_{\tau^\ell_j}\Big]| \le \mathbb{E}_{r_j}\Big[\beta\mathbf{1}_{\tau^\ell_{j-1}<\tau^\ell_j=T}\Big] + \mathbb{E}_{r_j}\Big[\beta(\tau^\ell_j - \tau^\ell_{j-1})\Big]^{\frac{1}{2}} ,$$

for $j < \kappa$.

This allows us to prove that

Proposition 4.4. *If* $(\mathbf{H}f)$*,* $(\mathbf{H}x2)$ *and* $(\mathbf{H}b3)$ *hold, then*

$$||Z^{\Re} - P^\pi Z^{\Re}||^2_{\mathcal{H}^2} \quad\le\quad C_L\kappa^{\frac{1}{2}}|\pi| .$$

Proof. 1. A similar approximation argument as the one used in step 2. of the proof of Proposition 4.2 allows to reduce our study to the case where $(\mathbf{H}r)$ and

- σ and b are C^2_b (smooth version of $(\mathbf{H}x2)$).
- h and l are C^3_b (smooth version of $(\mathbf{H}b3)$).

2. Under $(\mathbf{H}f)$, Remark 3.3 implies that,

$$V^{\ell,j}_{s,t} \quad=\quad \eta_s\mathbb{E}_t\Big[A^{\ell,j}_s\Big] \ ,\ \ell\in\{1,\ldots,d\}\ ,\ s\le t\in[t_i,t_{i+1})\subset[r_j,r_{j+1}] ,$$

where

$$\begin{aligned}
\eta_s &:= (\Lambda^\ell_s\nabla X_s)^{-1}\sigma(X_s)\\
A^{\ell,j}_s &:= \nabla\phi^\ell_{\tau^\ell_j}\Lambda^\ell_{\tau_j}\nabla X_{\tau_j} + \int_s^{\tau^\ell_j}\Big(\nabla_x f^\ell(\Theta^{\Re}_u)\Lambda^\ell_u\nabla X_u + \nabla_y f^\ell(\Theta^{\Re}_u)\Lambda^\ell_u\nabla\widetilde{Y}^{\Re}_u\Big)\,\mathrm{d}u.
\end{aligned}$$

Recall (4.3) in the proof of Proposition 4.3. We then have to study the quantities

$$\sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[|V^{\ell,j}_{t,t} - V^{\ell,j}_{t_k,t}|^2\Big]\,\mathrm{d}t \ \text{ and }\ \sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[|V^{\ell,j}_{t_k,t} - V^{\ell,j}_{t_k,t_k}|^2\Big]\,\mathrm{d}t. \tag{4.9}$$

By (4.7) in the proof of Proposition 4.3 applied under $(\mathbf{H}f)$ (i.e. $\alpha(\kappa)=1$), we first obtain that

$$\sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[|V^{\ell,j}_{t,t} - V^{\ell,j}_{t_k,t}|^2\Big]\,\mathrm{d}t \quad\le\quad C_L|\pi| .$$

To control the second term, we can reproduce line by line the arguments used in the proof of Proposition 5.2 in [3] to obtain

$$\sum_{j=0}^{\kappa-1}\sum_{k=i_j}^{i_{j+1}-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[|V^{\ell,j}_{t_k,t} - V^{\ell,j}_{t_k,t_k}|^2\Big]\,\mathrm{d}t \quad\le\quad C_L|\pi|(1+\Sigma^\ell + (\tilde{\Sigma}^\ell)^{\frac{1}{2}}) \tag{4.10}$$

where

$$\Sigma^\ell := \sum_{j=1}^{\kappa-1} \mathbb{E}\Big[|V^{\ell,j-1}_{r_j,r_j}|^2 - |V^{\ell,j}_{r_j,r_j}|^2\Big] \ \text{ and } \ \tilde{\Sigma}^\ell \ := \ \sum_{j=1}^{\kappa-1} \mathbb{E}\Big[|A^{\ell,j-1}_{r_j}|^4 - |A^{\ell,j}_{r_j}|^4\Big]\,.$$

3. We now study Σ^ℓ and $\tilde{\Sigma}^\ell$. Using (3.12), (3.15) and (3.19), we first get that

$$|V^{\ell,j-1}_{r_j,r_j}|^2 - |V^{\ell,j}_{r_j,r_j}|^2 \le \beta\left(|\mathbb{E}_{r_j}\Big[\tau^\ell_j - \tau^\ell_{j-1}\Big]| + |\mathbb{E}_{r_j}\Big[\nabla\phi^\ell_{\tau^\ell_{j-1}}\Lambda^\ell_{\tau^\ell_{j-1}}\nabla X_{\tau^\ell_{j-1}} - \nabla\phi^\ell_{\tau^\ell_j}\Lambda^\ell_{\tau^\ell_j}\nabla X_{\tau^\ell_j}\Big]|\right),$$

which, by Lemma 4.1, implies

$$\Sigma^\ell \ \le \ C_L\,\kappa^{\frac{1}{2}}\ .$$

Similar arguments lead to

$$\tilde{\Sigma}^\ell \ \le \ C_L\,\kappa^{\frac{1}{2}}\ .$$

We conclude the proof by plugging these estimates in (4.10). □

Proof of Lemma 4.1. 1. For all $\ell \in \{1\ldots d\}$, $j<\kappa$, we introduce:

$$\begin{aligned}
\Delta\phi^\ell_j \ &:= \ \nabla\phi^\ell_{\tau^\ell_{j-1}}\Lambda^\ell_{\tau^\ell_{j-1}}\nabla X_{\tau^\ell_{j-1}} - \nabla\phi^\ell_{\tau^\ell_j}\Lambda^\ell_{\tau^\ell_j}\nabla X_{\tau^\ell_j}\ ,\\
\Delta h^\ell_j \ &:= \ \nabla h^\ell(X_{\tau^\ell_{j-1}})_{\tau^\ell_{j-1}}\Lambda^\ell_{\tau^\ell_{j-1}}\nabla X_{\tau^\ell_{j-1}} - \nabla h^\ell(X_{\tau^\ell_j})\Lambda^\ell_{\tau^\ell_j}\nabla X_{\tau^\ell_j}\ ,\\
\Delta l^\ell_j \ &:= \ \nabla l^\ell(X_{\tau^\ell_{j-1}})_{\tau^\ell_{j-1}}\Lambda^\ell_{\tau^\ell_{j-1}}\nabla X_{\tau^\ell_{j-1}} - \nabla l^\ell(X_{\tau^\ell_j})\Lambda^\ell_{\tau^\ell_j}\nabla X_{\tau^\ell_j}\ ,
\end{aligned}$$

Since

$$\nabla\phi^\ell_{\tau^\ell_{j-1}}\Lambda^\ell_{\tau^\ell_{j-1}}\nabla X_{\tau^\ell_{j-1}} - \nabla\phi^\ell_{\tau^\ell_j}\Lambda^\ell_{\tau^\ell_j}\nabla X_{\tau^\ell_j} \ = \ \Delta\phi^\ell_j\left(\mathbf{1}_{\{\tau^\ell_{j-1}<\tau^\ell_j<T\}} + \mathbf{1}_{\{\tau^\ell_{j-1}<\tau^\ell_j=T\}}\right),$$

it follows from (3.15) and (3.12) that

$$|\mathbb{E}_{r_j}\Big[\Delta\phi^\ell_j\Big]| \le \mathbb{E}_{r_j}\Big[\beta\mathbf{1}_{\{\tau^\ell_{j-1}<\tau^\ell_j=T\}}\Big] + \mathbb{E}_{r_j}\Big[|\Delta\phi^\ell_j|\mathbf{1}_{\{\tau^\ell_{j-1}<\tau^\ell_j<T\}}\Big]\ . \tag{4.11}$$

2. We now fix a coordinate $\ell \in \{1,\ldots,d\}$ and set $U^\ell_j := \{\tau^\ell_{j-1} < \tau^\ell_j < T\}$,

$$\Delta Y^\ell_j = |(\widetilde{Y}^{\Re}_{\tau^\ell_{j-1}})^\ell - (Y^{\Re}_{\tau^\ell_j})^\ell| \ \text{ and } \ \Delta X_j = |X_{\tau^\ell_{j-1}} - X_{\tau^\ell_j}|\ .$$

Using the same arguments as in the proof of Proposition 4.1, we obtain

$$\mathbb{E}_{r_j}\Big[|\Delta Y^\ell_j|^2 + |\Delta X_j|^2\Big] \le \mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})]\,. \tag{4.12}$$

Since h^ℓ and l^ℓ are L-Lipschitz continuous and $h^\ell \ge l^\ell + \epsilon$, we can find $\eta^\ell > 0$ and $\epsilon^\ell > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$:

$$|x_1 - x_2| \le \eta^\ell \Longrightarrow h(x_1) - l(x_2) > \epsilon^\ell. \tag{4.13}$$

Observe that by choosing L large enough we can assume that $\frac{1}{\epsilon} \leq L$ so that $\frac{1}{\eta^\ell} + \frac{1}{\epsilon^\ell} \leq C_L$.
We then introduce the three following disjoint sets of $\mathcal{F}_T$

$$\begin{cases} A_j^\ell &= \{|\Delta Y_j^\ell| \leq \epsilon^\ell, |\Delta X_j| \leq \eta^\ell\} \cap U_j^\ell \\ B_j^\ell &= \{|\Delta Y_j^\ell| \leq \epsilon^\ell, |\Delta X_j| > \eta^\ell\} \cap U_j^\ell \\ C_j^\ell &= \{|\Delta Y_j^\ell| > \epsilon^\ell\} \cap U_j^\ell \end{cases}$$

Clearly, $A_j^\ell \cup B_j^\ell \cup C_j^\ell = U_j^\ell$.
3.a On $A_j^\ell \cap \{(Y^{\Re}_{\tau_j^\ell})^\ell = h^\ell(X_{\tau_j^\ell})\}$, we have $(Y^{\Re}_{\tau_j^\ell})^\ell - l^\ell(X_{\tau_{j-1}^\ell}) > \epsilon^\ell$, by (4.13). But on A_j^ℓ, we also have $|(Y^{\Re}_{\tau_j^\ell})^\ell - (\widetilde{Y}^{\Re}_{\tau_{j-1}^\ell})^\ell| \leq \epsilon^\ell$, thus $(Y^{\Re}_{\tau_{j-1}^\ell})^\ell = h^\ell(X_{\tau_{j-1}^\ell})$. Using the same arguments on $A_j^\ell \cap \{(Y^{\Re}_{\tau_j^\ell})^\ell = l^\ell(X_{\tau_j^\ell})\}$, we obtain $(Y^{\Re}_{\tau_{j-1}^\ell})^\ell = l^\ell(X_{\tau_j^\ell})$. And, since $\left(A_j^\ell \cap \{(Y^{\Re}_{\tau_j^\ell})^\ell = h^\ell(X_{\tau_j^\ell})\}\right) \cup \left((A_j^\ell \cap \{(Y^{\Re}_{\tau_j^\ell})^\ell = l^\ell(X_{\tau_j^\ell})\}\right) = A_j^\ell$, we have

$$\mathbb{E}_{r_j}\Big[|\Delta\phi_j^\ell|\mathbf{1}_{U_j^\ell}\Big] \leq \mathbb{E}_{r_j}\Big[(|\Delta h_j^\ell| + |\Delta l_j^\ell|)\mathbf{1}_{A_j^\ell}\Big] + \mathbb{E}_{r_j}\Big[|\Delta\phi_j^\ell|\Big]\,(\mathbf{1}_{B_j^\ell} + \mathbf{1}_{C_j^\ell}). \tag{4.14}$$

Using ($\mathbf{H}b3$), we have

$$\mathbb{E}_{r_j}\Big[(|\Delta h_j^\ell| + |\Delta l_j^\ell|)\mathbf{1}_{A_j^\ell}\Big] \leq \mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})]^{\frac{1}{2}}$$

and, by Tchebytchev's inequality and (4.12),

$$\mathbb{E}_{r_j}\Big[|\Delta\phi_j^\ell|\Big]\,(\mathbf{1}_{B_j^\ell} + \mathbf{1}_{C_j^\ell}) \leq \mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})]^{\frac{1}{2}} \;.$$

Using (4.14), this leads to

$$\mathbb{E}_{r_j}\Big[|\Delta\phi_j^\ell|\mathbf{1}_{U_j^\ell}\Big] \leq \mathbb{E}_{r_j}[\beta(\tau_j - \tau_{j-1})]^{\frac{1}{2}} \;,$$

which concludes the proof. □

5 Discrete time approximation of discretely RBSDEs

As an application of the regularity results stated in the last section, we now study the convergence of an Euler scheme approximation method for discretely reflected BSDEs. Using an approximation argument, we will then propose an extension of this method to continuously reflected BSDEs in the next section.

5.1 Discrete time approximation of the forward process

As in the previous section, we consider a grid $\pi = \{0 =: t_0 < t_1 < \ldots < t_n := T\}$ of the time interval $[0, T]$ with modulus $|\pi|$, such that $\Re \subset \pi$.

As usual, X is approximated by its Euler scheme X^π defined by:

$$\begin{cases} X_0^\pi &= X_0 \\ X_{t_{i+1}}^\pi &= X_{t_i}^\pi + b(X_{t_i}^\pi)(t_{i+1}-t_i) + \sigma(X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i}) \ , \ i \le n-1 \ , \end{cases}$$

and for $t \in [t_i, t_{i+1})$, $i \le n-1$,

$$X_t^\pi \ = \ X_{t_i}^\pi + b(X_{t_i}^\pi)(t-t_i) + \sigma(X_{t_i}^\pi)(W_t - W_{t_i}) \ .$$

Under ($\mathbf{H}x1$), b and σ are L-Lipschitz continuous, thus we have (see e.g. [13])

$$\| \sup_{t\le T} |X_t - X_t^\pi| \, \|_{L^p} + \max_{i<n} \| \sup_{t\in[t_i,t_{i+1}]} |X_t - X_{t_i}^\pi| \, \|_{L^p} \ \le \ C_L^p \, |\pi|^{\frac{1}{2}} \ , \ p \ge 1 \ . \quad (5.1)$$

5.2 Euler Scheme for discretely reflected BSDEs

We now introduce a discrete-time approximation scheme for the discretely reflected BSDE of the form

$$\begin{cases} \bar{Z}_{t_i}^\pi &= (t_{i+1}-t_i)^{-1} \, \mathbb{E}_{t_i}\Big[(W_{t_{i+1}} - W_{t_i})(Y_{t_{i+1}}^\pi)'\Big] \\ \widetilde{Y}_{t_i}^\pi &= \mathbb{E}_{t_i}\Big[Y_{t_{i+1}}^\pi\Big] + (t_{i+1}-t_i) f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \\ Y_{t_i}^\pi &= \mathcal{R}(t_i, X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi) \ , \ i \le n-1 \ , \end{cases} \quad (5.2)$$

with terminal condition

$$\widetilde{Y}_T^\pi \quad = \quad Y_T^\pi := g(X_T^\pi) \ .$$

This kind of backward scheme has been already considered when no reflection occurs, see e.g. [5], and in the simply reflected case, see e.g. [3, 15] and the references therein.

Combining an induction argument with the Lispchitz-continuity of g, f and the projection operator, one easily checks that the above processes are square integrable and that the conditional expectations are well defined at each step of the algorithm.

For later use, we introduce the continuous time scheme associated to $(Y^\pi, \bar{Z}^\pi)$. By the martingale representation theorem, there exists $Z^\pi \in \mathcal{H}^2(\mathbb{M}^d)$ such that

$$Y_{t_{i+1}}^\pi = \mathbb{E}_{t_i}\Big[Y_{t_{i+1}}^\pi\Big] + \int_{t_i}^{t_{i+1}} (Z_u^\pi)' \mathrm{d}W_u \ , \quad i \le n-1 \ .$$

We then define $\widetilde{Y}^\pi$ on $[t_i, t_{i+1})$ by

$$\widetilde{Y}_t^\pi \quad = \quad Y_{t_{i+1}}^\pi + (t_{i+1}-t) f(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) - \int_t^{t_{i+1}} (Z_u^\pi)' \mathrm{d}W_u \ , \quad (5.3)$$

and set

$$Y_t^\pi \quad := \quad \mathcal{R}(t, X_t^\pi, \widetilde{Y}_t^\pi) \ , \quad \text{for} \quad t \le T \ .$$

Remark that, by the Itô isometry,

$$\bar{Z}^\pi \quad = \quad P^\pi Z^\pi \ , \quad (5.4)$$

where P^π is defined in (4.2).

5.3 Convergence results

We first provide estimates on the difference between $(Y^{\Re}, Z^{\Re})$ and $(Y^{\pi}, \bar{Z}^{\pi})$.

Proposition 5.1. *Assume that* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$ *hold, then*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t^{\Re} - Y_t^{\pi}|^2\Big] + ||Z^{\Re} - \bar{Z}^{\pi}||_{\mathcal{H}^2}^2 \;\leq\; C_L\Big(||\widetilde{Y}^{\Re} - \mathcal{D}^{\pi}\widetilde{Y}^{\Re}||_{\mathcal{H}^2}^2 + ||Z^{\Re} - P^{\pi}Z^{\Re}||_{\mathcal{H}^2}^2 + \kappa\mathbb{E}\Big[\max_{r\in\Re}|X_r - X_r^{\pi}|^2\Big] + ||X - \mathcal{D}^{\pi}X^{\pi}||_{\mathcal{S}^2}^2\Big)\;.$$

Moreover, if f^{ℓ} depends on (y, z) only through $(y^{\ell}, z^{\cdot\ell})$, we have

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t^{\Re} - Y_t^{\pi}|^2\Big] \;\leq\; C_L\Big(||\widetilde{Y}^{\Re} - \mathcal{D}^{\pi}\widetilde{Y}^{\Re}||_{\mathcal{H}^2}^2 + ||Z^{\Re} - P^{\pi}Z^{\Re}||_{\mathcal{H}^2}^2 + ||X - \mathcal{D}^{\pi}X^{\pi}||_{\mathcal{S}^2}^2\Big)\,.$$

Before providing the proof of this result, let us observe that combining it with Proposition 4.1, Proposition 4.3, Proposition 4.4 and (5.1), we obtain an upper bound on the approximation error between the Euler scheme (5.2) and the discretely reflected BSDE (2.4).

Theorem 5.1. *Set* $(\alpha(\kappa), \gamma(\kappa)) = (\kappa^2, \kappa)$ *under* $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $(\alpha(\kappa), \gamma(\kappa)) = (\kappa, 1)$ $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$ *and* $(\alpha(\kappa), \gamma(\kappa)) = (\kappa, 0)$ *under* $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$ *then the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - (\mathcal{D}^{\pi}\widetilde{Y}^{\pi})_t|^2\Big] + ||Z^{\Re} - \bar{Z}^{\pi}||_{\mathcal{H}^2}^2 \leq C_L\Big(\alpha(\kappa)|\pi| + \gamma(\kappa)|\pi|^{\frac{1}{2}}\Big)\,.$$

Moreover if $(\mathbf{H}x2)$-$(\mathbf{H}b3)$ *hold and f^{ℓ} depends on (y, z) only through $(y^{\ell}, z^{\cdot\ell})$, then we have*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - (\mathcal{D}^{\pi}\widetilde{Y}^{\pi})_t|^2\Big] \;\leq\; C_L\kappa^{\frac{1}{2}}|\pi|\;.$$

Remark 5.1. The estimates above are stated in a fairly general setting. They can be improved in some particular cases.

1. If $X = X^{\pi}$ on π, i.e. X is "perfectly simulated", then the term $\mathbb{E}\big[\max_{r\in\Re}|X_r - X_r^{\pi}|^2\big] = 0$ disappears in the estimate of Proposition 5.1. In particular, if $(\mathbf{H}x2)$-$(\mathbf{H}b3)$ hold and f^{ℓ} depends on (y, z) only through $(y^{\ell}, z^{\cdot\ell})$, then we have

$$||Z^{\Re} - \bar{Z}^{\pi}||_{\mathcal{H}^2}^2 \leq C_L\kappa^{\frac{1}{2}}|\pi|.$$

2. If f does not depend on z, then

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - (\mathcal{D}^{\pi}\widetilde{Y}^{\pi})_t|^2\Big] \;\leq\; C_L|\pi|\;.$$

This follows from the fact that, in this case, the term $\int_{t_{i-1}}^{t_i}(|Z_u^{\Re} - \bar{Z}_{t_{i-1}}^{\Re}|^2)\mathrm{d}u$ in (5.6) below disappears.

The proof of Proposition 5.1 relies on the following Remark.

Remark 5.2. Under $(\mathbf{H}b1)$, for $t \in \pi$ and each $\ell \in \{1, \ldots, d\}$, there exists S_t^ℓ, Q_t^ℓ in $\mathcal{F}_t$ such that $S_t^\ell \cap Q_t^\ell = \emptyset$ and

$$|(Y_t^{\Re})^\ell - (Y_t^\pi)^\ell|^2 \quad \leq \quad |(\widetilde{Y}_t^{\Re})^\ell - (\widetilde{Y}_t^\pi)^\ell|^2 \mathbf{1}_{S_t^\ell} + C_L |X_t - X_t^\pi|^2 \mathbf{1}_{Q_t^\ell} \; .$$

This is shown by arguing as in the proof of Lemma 2.1.
Moreover, for $t \in [0,T] \setminus \Re$, we have $|(Y_t^{\Re})^\ell - (Y_t^\pi)^\ell| = |(\widetilde{Y}_t^{\Re})^\ell - (\widetilde{Y}_t^\pi)^\ell|$ and for $t \in \pi \setminus \Re$, we can set $S_t^\ell = \Omega$ and $Q_t^\ell = \emptyset$.

Proof of Proposition 5.1. We adapt the proof of Theorem 3.1 in [5] to our context.
1.a We set $\delta Y = Y^{\Re} - Y^\pi$, $\delta \widetilde{Y} = \widetilde{Y}^{\Re} - \widetilde{Y}^\pi$, $\delta Z = Z^{\Re} - \bar{Z}^\pi$ and $\delta X = X - X^\pi$. Observe that, by (5.4) and Jensen's inequality,

$$\mathbb{E}\Big[|\bar{Z}_t^{\Re} - \bar{Z}_t^\pi|^2\Big] \quad \leq \quad (t_{i+1} - t_i)^{-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\Big[|Z_u^{\Re} - Z_u^\pi|^2\Big] \, \mathrm{d}u \; ,$$

where $\bar{Z}^{\Re} = P^\pi Z^{\Re}$.
Applying Itô's formula to $|\delta \widetilde{Y}|^2$ on $[t_i, t_{i+1}) \subset [r_j, r_{j+1})$, using the last inequality and standard arguments (see e.g. step 1. of Proposition 7.1 in the Appendix), we obtain for all $s \leq t_i$

$$\begin{aligned} \mathbb{E}_s\Big[|\delta \widetilde{Y}_t|^2 + \int_t^{t_{i+1}} |\delta Z_u|^2 \mathrm{d}u\Big] \quad \leq \quad & \mathbb{E}_s\Big[|\delta Y_{t_{i+1}}|^2 + \alpha \int_t^{t_{i+1}} |\delta \widetilde{Y}_u|^2 \mathrm{d}u + C_L B_{i+1} \\ & + \frac{C_L}{\alpha}\Big(|t_{i+1} - t_i||\delta \widetilde{Y}_{t_i}|^2 + \int_{t_i}^{t_{i+1}} |\delta Z_u|^2 \mathrm{d}u\Big)\Big] \end{aligned} \tag{5.5}$$

where $\alpha > 1$ is to be chosen later on and for $i \in \{1, \ldots, n\}$

$$B_i \quad := \quad \int_{t_{i-1}}^{t_i} (|X_u - X_{t_{i-1}}^\pi|^2 + |\widetilde{Y}_u^{\Re} - \widetilde{Y}_{t_{i-1}}^{\Re}|^2 + |Z_u^{\Re} - \bar{Z}_{t_{i-1}}^{\Re}|^2) \mathrm{d}u \; . \tag{5.6}$$

By Gronwall's Lemma, we deduce that, for all $t \in [t_i, t_{i+1})$,

$$\mathbb{E}_s\Big[|\delta \widetilde{Y}_t|^2\Big] \leq e^{\alpha C_L |t_{i+1} - t_i|} \mathbb{E}_s\Big[|\delta Y_{t_{i+1}}|^2 + C_L B_{i+1} + \frac{C_L}{\alpha}(|t_{i+1} - t_i||\delta \widetilde{Y}_{t_i}|^2 + \int_{t_i}^{t_{i+1}} |\delta Z_u|^2 \mathrm{d}u)\Big] \,. \tag{5.7}$$

Combining the last equation with (5.5), choosing α such that $C_L/\alpha \leq 1/4$ and then working with $|\pi|$ small enough such that $\alpha|\pi| e^{C_L \alpha |\pi|} \leq 2\alpha|\pi| \leq 1$, we compute that

$$\mathbb{E}_s\Big[|\delta \widetilde{Y}_{t_i}|^2 + \frac{1}{2} \int_{t_i}^{t_{i+1}} |\delta Z_u|^2 \mathrm{d}u\Big] \quad \leq \quad e^{C_L |t_{i+1} - t_i|} \mathbb{E}_s\big[|\delta Y_{t_{i+1}}|^2 + C_L B_{i+1}\big] \; . \tag{5.8}$$

1.b For $j \leq \kappa$, we define i_j through $t_{i_j} = r_j$. Since $|\delta Y_t| = |\delta \widetilde{Y}_t|$ for all $t \in \pi \setminus \Re$, we deduce from (5.8) and an induction argument that, for $i \in [i_j, i_{j+1})$,

$$\mathbb{E}\Big[|\delta \widetilde{Y}_{t_i}|^2\Big] \leq e^{C_L |r_{j+1} - t_i|} \mathbb{E}\left[|\delta Y_{r_{j+1}}|^2 + C_L \sum_{k=i_j+1}^{i_{j+1}} B_k\right] \; . \tag{5.9}$$

Summing up over i in (5.8), we also obtain

$$\mathbb{E}\Big[\int_{r_j}^{r_{j+1}}|\delta Z_u|^2 \mathrm{d}u\Big] \leq C_L \mathbb{E}\Big[|\delta X_{r_{j+1}}|^2+|\delta \widetilde{Y}_{r_{j+1}}|^2-|\delta \widetilde{Y}_{r_j}|^2+|\pi|\sum_{k=i_j+1}^{i_{j+1}}|\delta \widetilde{Y}_{t_k}|^2+\sum_{k=i_j+1}^{i_{j+1}} B_k\Big] .$$

Summing up over j, this leads to

$$\mathbb{E}\Big[\int_0^T |\delta Z_u|^2 \mathrm{d}u\Big] \leq C_L \sup_{t\in\pi} \mathbb{E}\Big[|\delta \widetilde{Y}_t|^2 + \sum_{i=1}^{n} B_i\Big] + \kappa \max_{r\in\Re} \mathbb{E}\big[|\delta X_r|^2\big] . \tag{5.10}$$

Using Remark 5.2, (5.9) and an induction argument we then obtain

$$\mathbb{E}\Big[|\delta \widetilde{Y}_{r_j}|^2 + |\delta Y_{r_j}|^2\Big] \leq C_L \mathbb{E}\Big[|\delta X_T|^2 + \kappa \max_{r\in\Re} |\delta X_r|^2 + \sum_{q=0}^{\kappa-1}\sum_{k=i_q+1}^{i_{q+1}} B_k\Big] , \; j<\kappa ,$$

which combined with (5.9) leads to

$$\sup_{i\leq n} \mathbb{E}\Big[|\delta \widetilde{Y}_{t_i}|^2 + |\delta Y_{t_i}|^2\Big] \leq C_L \mathbb{E}\Big[\kappa \max_{r\in\Re} |\delta X_r|^2 + \sum_{i=1}^{n} B_i\Big] . \tag{5.11}$$

The proof is then concluded by plugging (5.11) in (5.10) and then combining (5.7) with (5.10) and (5.11).

2. We now turn to the case where f^ℓ depends on (y,z) only through $(y^\ell, z^{\cdot\ell})$.
In this case (5.7) and (5.8) reads

$$\begin{aligned}\mathbb{E}_s\Big[|(\delta \widetilde{Y}_t)^\ell|^2\Big] \;\leq\; & e^{\alpha C_L|t_{i+1}-t_i|}\mathbb{E}_s\Big[|(\delta Y_{t_{i+1}})^\ell|^2 + C_L B_i \\ & + \frac{C_L}{\alpha}(|t_{i+1}-t_i||(\delta \widetilde{Y}_{t_i})^\ell|^2 + \int_{t_i}^{t_{i+1}}|(\delta Z_u)^\ell|^2 \mathrm{d}u)\Big] .\end{aligned} \tag{5.12}$$

and

$$\mathbb{E}_s\Big[|(\delta \widetilde{Y}_{t_i})^\ell|^2 + \frac{1}{2}\int_{t_i}^{t_{i+1}}|(\delta Z_u)^\ell|^2 \mathrm{d}u\Big] \leq e^{C_L|t_{i+1}-t_i|}\mathbb{E}_s\Big[|(\delta Y_{t_{i+1}})^\ell|^2 + C_L B_i\Big] , \tag{5.13}$$

for $t\in[t_i,t_{i+1})$, $s\leq t_i$, $i<n$.
For each $\ell\in\{1,\ldots,d\}$ and $i<n-1$, we then introduce the sequences of sets U^ℓ and $\tilde{U}^\ell$ defined by

$$\begin{cases} U_i^\ell := \Omega \;\text{ and }\; U_{i+k}^\ell := U_{i+k-1}^\ell \cap S_{t_{i+k}}^\ell , \\ \tilde{U}_i^\ell := \emptyset \;\text{ and }\; \tilde{U}_{i+k}^\ell := U_{i+k-1}^\ell \cap Q_{t_{i+k}}^\ell , \end{cases}$$

for $k\in[1,n-i-1]$.
Recall the definition of S^ℓ and Q^ℓ in Remark 5.2. Since $S_t^\ell \cap Q_t^\ell = \emptyset$ for each t of π, we have $U_{i+k}^\ell \cap \tilde{U}_{i+k}^\ell = \emptyset$ and $\tilde{U}_{i+k}^\ell \cap \tilde{U}_{i+j}^\ell = \emptyset$, for all $k\in[1,n-i-1]$, $j\in[k+1,n-i-1]$. Moreover, $U_{i+k}^\ell, \tilde{U}_{i+k}^\ell \in \mathcal{F}_{t_{i+k}}$.

Using (5.13), Remark 5.2 and an induction argument, we deduce that, for $k \in [1, n-i-1]$,

$$\mathbb{E}_{t_i}\Big[|(\delta\widetilde{Y}_{t_i})^\ell|^2\Big] \le C_L \mathbb{E}_{t_i}\left[|(\delta\widetilde{Y}_{t_{i+k+1}})^\ell|^2 + \sum_{j=1}^{k}(|\delta X_{t_{i+j}}|^2 \mathbf{1}_{\tilde{U}^\ell_{i+j}} + B_{i+j})\right]$$

In particular, for $k = n-i-1$, this leads to

$$\mathbb{E}_{t_i}\Big[|(\delta\widetilde{Y}_{t_i})^\ell|^2\Big] \le C_L \mathbb{E}_{t_i}\left[\max_{r\in\Re}|\delta X_r|^2 + \sum_{j=i+1}^{n} B_j\right]$$

since $\sum_{i=1}^{n-j-1}\mathbf{1}_{\tilde{U}^\ell_{i+j}} \le 1$ and $|\delta Y_T| \le C_L|\delta X_T|$.
Combining the last inequality with (5.12), (5.13) and using Remark 5.2 again, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|(\delta\widetilde{Y}_t)^\ell|^2 + |(\delta Y_t)^\ell|^2\Big] \quad \le \quad C_L \mathbb{E}\left[\max_{r\in\Re}|\delta X_r|^2 + \sum_{i=1}^{n} B_i\right] \ .$$

The proof is then concluded by summing up over ℓ. □

6 Extensions to continuously reflected BSDEs

We now apply the results of the last section to continuously reflected BSDEs.
We first obtain a regularity results for the solution of such equation in the spirit of [15]. We then show that the Euler scheme (5.2) can be used to approximate continuously reflected BSDE, provided that $\Re$ and π are conveniently chosen.

In this section, we assume the existence and uniqueness of a strong solution to the continuously reflected BSDE defined by

$$\begin{cases} Y^\ell_t = g^\ell(X_T) + \int_t^T f^\ell(X_u, Y_u, Z_u)\mathrm{d}u - \int_t^T Z^\ell_u \mathrm{d}W_u + \int_t^T \mathrm{d}K^{\ell\,+}_u - \int_t^T \mathrm{d}K^{\ell\,-}_u \\ l^\ell(X_t) \le Y^\ell_t \le h^\ell(X_t) \ , \ \forall t \in [0,T] \ , \ a.s. \\ \int_0^T (Y^\ell_s - l^\ell(X_s))\mathrm{d}K^{\ell\,+}_s = \int_0^T (Y^\ell_s - h^\ell(X_s))\mathrm{d}K^{\ell\,-}_s = 0, \end{cases} \tag{6.1}$$

for each $\ell \in \{1,\dots,d\}$ and where $K^{\ell\,+}, K^{\ell\,-} \in \mathcal{S}^2(\mathbb{R})$ are continuous, increasing and $K^{\ell\,+}_0 = K^{\ell\,-}_0 = 0$.

Remark 6.1. 1. When $d = 1$ and l, h are C^1_b with L-Lipschitz continuous derivative and $h \ge l + \epsilon$, for some $\epsilon > 0$, existence and uniqueness to the above equations are well known, see e.g. [7]. Obviously this immediately extends to the case $d > 1$ whenever f^ℓ depends on (y, z) through (y^ℓ, z^ℓ) only.
2. When $d \ge 2$ and h, l are constant, existence and uniqueness follow from [8].

The Proposition below will allow us to extend the results of the last section to continuously reflected BSDE. Roughly speaking, it means that $(Y^{\Re}, Z^{\Re})$ is a good approximation for (Y, Z).

Proposition 6.1. *Set $q = \frac{1}{2}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$ and $q = 1$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b2)$, then we have*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - Y_t^{\Re}|^2\Big] + \sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}_t^{\Re}|^2\Big] + ||Z - Z^{\Re}||^2_{\mathcal{H}^2} \;\; \leq \;\; C_L|\Re|^q ,$$

where $|\Re|$ is the modulus of $\Re$.

Proof. First, observe that we can consider each coordinate separately. We can then follow essentially the same arguments as in the proof of Proposition 4.1 in [3]. In particular, we have to control both

$$\int_t^{r_{j+1}} (l^\ell(X_s) - (\widetilde{Y}_s^{\Re})^\ell)\mathrm{d}K_s^{\ell+} \quad \text{and} \quad \int_t^{r_{j+1}} ((\widetilde{Y}_s^{\Re})^\ell - h^\ell(X_s))\mathrm{d}K_s^{\ell-} \,,\; \ell \in \{1,\ldots,d\}\,.$$

For all $s \leq T$, we have

$$l^\ell(X_s) - (\widetilde{Y}_s^{\Re})^\ell \;\; \leq \;\; \mathbb{E}_s\Big[l^\ell(X_s) - l^\ell(X_{r_{j+1}}) + \int_s^{r_{j+1}} |f^\ell(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re})|\mathrm{d}u\Big] \,, \tag{6.2}$$

$$(\widetilde{Y}_s^{\Re})^\ell - h^\ell(X_s) \;\; \leq \;\; \mathbb{E}_s\Big[h^\ell(X_{r_{j+1}}) - h^\ell(X_s) + \int_s^{r_{j+1}} |f^\ell(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re})|\mathrm{d}u\Big] \,. \tag{6.3}$$

Under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b2)$, the control on h and l given by the assumption and the Lipschitz-continuity of σ, b and f, implies that,

$$\begin{aligned} l^\ell(X_s) - (\widetilde{Y}_s^{\Re})^\ell \;\; &\leq \;\; C_L\mathbb{E}_s\Big[\int_s^{r_{j+1}} (1 + |\rho_1^\ell(X_s)'b(X_u)| + |\rho_3^\ell(X_s)|(1 + |X_u|^2))\mathrm{d}u\Big] \\ &\qquad + C_L\,\mathbb{E}_s\Big[\int_s^{r_{j+1}} (|X_u| + |\widetilde{Y}_u^{\Re}| + |Z_u^{\Re}|)\mathrm{d}u\Big] \end{aligned}$$

It then follows from Cauchy-Schwartz inequality and Propositions 2.1 and 4.2 that

$$l^\ell(X_s) - (\widetilde{Y}_s^{\Re})^\ell \;\; \leq \;\; |\Re|\,\beta\,.$$

Similar arguments applied to (6.3) lead to

$$(\widetilde{Y}_s^{\Re})^\ell - h^\ell(X_s) \;\; \leq \;\; |\Re|\,\beta\,.$$

Under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, we use the Lipschitz-continuity of l, to obtain

$$l^\ell(X_s) - (\widetilde{Y}_s^{\Re})^\ell \;\; \leq \;\; C_L\mathbb{E}_s\Big[L|X_s - X_{r_{j+1}}| + \int_s^{r_{j+1}} (|X_u| + |\widetilde{Y}_u^{\Re}| + |Z_u^{\Re}|)\mathrm{d}u\Big]\,.$$

It follows then from Proposition 2.1, Proposition 4.2 and Cauchy-Schwartz inequality that

$$\begin{aligned} l^\ell(X_s) - (\widetilde{Y}_s^{\Re})^\ell \;\; &\leq \;\; (|\Re|^{\frac{1}{2}} + \kappa^{\frac{1}{2}}|\Re|)\,\beta \\ &\leq \;\; |\Re|^{\frac{1}{2}}\,\beta\,. \end{aligned}$$

Similarly, we have

$$(\widetilde{Y}_s^{\Re})^\ell - h^\ell(X_s) \quad \leq \quad |\Re|^{\frac{1}{2}}\,\beta\ .$$

In both cases, the proof is then concluded by arguing exactly in [3]. $\square$

Combining this Proposition with Proposition 4.1, and Proposition 4.3, we deduce the following regularity property for (Y, Z).

Corollary 6.1. *Set $q = \frac{1}{3}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $q = \frac{1}{2}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$ and $q = 1$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b2)$, then the following holds*

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t - (\mathcal{D}^\pi Y)_t|^2\big] \leq C_L|\pi|^q \qquad \text{and} \qquad ||Z - P^\pi Z||^2_{\mathcal{H}^2} \leq C_L|\pi|^{\frac{q}{2}}\ .$$

Moreover, if $q = \frac{1}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$ and $q = \frac{2}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$, then we have

$$||Z - P^\pi Z||^2_{\mathcal{H}^2} \leq C_L|\pi|^q\ .$$

Proof. 1. We first study the regularity of Y. Since,

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t - (\mathcal{D}^\pi Y)_t|^2\big] \quad \leq \quad C_L\Big(\sup_{t\in[0,T]} \mathbb{E}\Big[|\widetilde{Y}_t^{\Re} - (\mathcal{D}^\pi \widetilde{Y}^{\Re})_t|^2\Big] + \sup_{t\in[0,T]} \mathbb{E}\Big[|Y_t - \widetilde{Y}_t^{\Re}|^2\Big]\Big)\ ,$$

the bound on $\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t - (\mathcal{D}^\pi Y)_t|^2\big]$ is obtained by applying Proposition 4.1 and Proposition 6.1, with $\Re$ and π chosen such that

$$|\Re| \leq \frac{C_L}{\kappa} \qquad \text{and} \qquad |\Re| = O(|\pi|^\alpha)\ , \tag{6.4}$$

with $\alpha = \frac{2}{3}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $\alpha = 1$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$ and $\alpha = \frac{1}{2}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b2)$.
2. We now turn to Z. By Jensens's inequality, we have

$$||Z - P^\pi Z||^2_{\mathcal{H}^2} \quad \leq \quad C_L(||Z^{\Re} - P^\pi Z^{\Re}||^2_{\mathcal{H}^2} + ||Z - Z^{\Re}||^2_{\mathcal{H}^2})\ .$$

Thus, choosing $\Re$ and π as in (6.4) with $\alpha = \frac{1}{3}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $\alpha = \frac{2}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $\alpha = \frac{1}{2}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b2)$, $\alpha = \frac{2}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$, we obtain the required bound by combining Proposition 4.3 with Proposition 6.1. $\square$

We now state the main result of this section which provides an upper bound for the convergence rate of the Euler scheme (5.2) to the continuously reflected BSDE (6.1).

Theorem 6.1. *Set $q = \frac{1}{6}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $q = \frac{1}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $q = \frac{1}{2}$ under $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$, then we have*

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t - (\mathcal{D}^\pi Y^\pi)_t|^2\big] + ||Z - \bar{Z}^\pi||^2_{\mathcal{H}^2} \quad \leq \quad C_L|\pi|^q\ .$$

Moreover, if $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$ *hold and* $X^\pi = X$ *on* π*, then*

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t - (\mathcal{D}^\pi Y^\pi)_t|^2\big] + ||Z - \bar{Z}^\pi||^2_{\mathcal{H}^2} \quad \leq \quad C_L|\pi|^{\frac{2}{3}} \ .$$

Proof. This is a direct consequence of Proposition 6.1 and Theorem 5.1 applied with $\Re$ and π defined as in (6.4), with $\alpha = \frac{1}{3}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $\alpha = \frac{2}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x1)$-$(\mathbf{H}b1)$, $\alpha = \frac{1}{2}$ under $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$ and $\alpha = \frac{2}{3}$ under $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$ and when $X^\pi = X$ on π. □

The results of the last Theorem can be compared to those of Theorem 4.1 in [3], which gives an upper bound for the rate of convergence in the case of unidimensional simply reflected BSDEs.
First, observe that $(\mathbf{H}b1)$ is weaker than the assumptions of Theorem 4.1 in [3] and the price to pay for these fairly mild regularity assumptions is the poor rate of convergence.
Second, under $(\mathbf{H}f)$-$(\mathbf{H}x2)$-$(\mathbf{H}b3)$, we are not able to retrieve the result of [3]. This can be explained by the structure of f in our multidimensional setting. In particular, its dependence with respect to all component of y prevents us to get rid of the term $\kappa\mathbb{E}\big[\max_{r\in\Re}|X_r - X_r^\pi|^2\big]$ in the first claim of Proposition 5.1 .

Let us conclude this paper with the following result dealing with the special case when the system of BSDE is decoupled:

Theorem 6.2. *Assume that* f^ℓ *depends on* (y,z) *only through* (y^ℓ, z^ℓ) *and set* $q = \frac{1}{2}$ *under* $(\mathbf{H}x1)$-$(\mathbf{H}b2)$*,* $q = \frac{2}{3}$ *under* $(\mathbf{H}x2)$-$(\mathbf{H}b3)$*, then we have*

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t - (\mathcal{D}^\pi Y^\pi)_t|^2\big] \quad \leq \quad C_L|\pi|^q \ .$$

Proof. This is an immediate consequence of Proposition 6.1 and the second claim of Theorem 5.1 applied with $\Re$ and π defined as in (6.4), with $\alpha = \frac{1}{2}$ under $(\mathbf{H}x1)$-$(\mathbf{H}b2)$ and $\alpha = \frac{2}{3}$ under $(\mathbf{H}x2)$-$(\mathbf{H}b3)$. □

Notice that, when $d = 1$, the last restriction on f trivially holds. In this case, Y can be interpreted as the price of a Game Option (see e.g. [6]). This provides an interesting financial application of our result.
Also, observe that, in Theorem 6.2, we obtain better bounds on the convergence rate. But, we are not able to retrieve the bounds of [3], due to the presence of two reflecting boundaries, see Lemma 4.1.

7 Appendix: *a priori* estimates

In this section we provide *a priori* estimates for reflected BSDEs in an abstract framework.

We consider processes $(Y^{\Re}, \widetilde{Y}^{\Re}, Z^{\Re}) \in \mathcal{S}^2(\mathbb{R}^d) \times \mathcal{S}^2(\mathbb{R}^d) \times \mathcal{H}^2(\mathbb{M}^d)$ such that:

$$\widetilde{Y}^{\Re}_t = Y^{\Re}_{r_{j+1}} + \int_t^{r_{j+1}} \hat{f}(u)\mathrm{d}u - \int_t^{r_{j+1}} (Z^{\Re}_u)'\mathrm{d}W_u \ , \ t \in [r_j, r_{j+1}) \ , j < \kappa \ , \tag{7.1}$$

where $\hat{f}$ is some adapted process satisfying

$$|\hat{f}| \le C_L(|\eta| + |\widetilde{Y}^{\Re}| + |Z^{\Re}|) \ , \text{ for some } \eta \in \mathcal{H}^2(\mathbb{R}) \ . \tag{7.2}$$

We also assume that

$$|\widetilde{Y}^{\Re}_t| \quad = \quad |Y^{\Re}_t| \ , \ \forall t \in [0,T] \setminus \Re \ , \tag{7.3}$$

and we work under the following assumption

- $(\mathbf{A}0)$: For all $\ell \in \{1, \dots, d\}$, $r \in \Re$,

  $$|(Y^{\Re}_r)^{\ell}| \quad \le \quad |(\widetilde{Y}^{\Re}_r)^{\ell}| \mathbf{1}_{S^{\ell}_r} + |\xi^{\ell}_r| \mathbf{1}_{Q^{\ell}_r}$$

  with $\xi \in \mathcal{S}^2(\mathbb{R})$, $S^{\ell}_r, Q^{\ell}_r \in \mathcal{F}_r$, $S^{\ell}_r \cap Q^{\ell}_r = \emptyset$ and $S^{\ell}_T = \emptyset$.

  Obviously, this implies that

  $$|Y^{\Re}_r|^2 \le |\widetilde{Y}^{\Re}_r|^2 + |\xi_r|^2 \ , \ r \in \Re \qquad \text{and} \qquad |Y^{\Re}_T|^2 \le |\xi_T|^2. \tag{7.4}$$

We shall also make use of the following assumption, which is a particular case of (7.2),
- $(\mathbf{A}f)$: For each $\ell \in \{1, ..., d\}$ and all $u \in [0,T]$, we have

$$|\hat{f}^{\ell}(u)| \quad \le \quad C_L(|\eta_u| + |\widetilde{Y}^{\Re}_u| + |(Z^{\Re}_u)^{\cdot \ell}|) \ .$$

In this framework, we can state the following proposition.

Proposition 7.1. *For all $s \le T$, the following holds*

$$\sup_{t \in [s,T]} \mathbb{E}_s\Big[|\widetilde{Y}^{\Re}_t|^2 + \int_t^T |Z^{\Re}_u|^2 \mathrm{d}u\Big] \le C_L \mathbb{E}_s\Big[|\widetilde{Y}^{\Re}_T|^2 + \kappa \max_{r \in \Re} |\xi_r|^2 + \int_0^T |\eta_u|^2 \mathrm{d}u\Big] \ .$$

When $(\mathbf{A}f)$ holds, we have

$$\sup_{t \in [s,T]} \mathbb{E}_s\Big[|\widetilde{Y}^{\Re}_t|^2\Big] \le C_L \mathbb{E}_s\Big[\max_{r \in \Re} |\xi_r|^2 + \int_0^T |\eta_u|^2 \mathrm{d}u\Big] \ , \ s \le T \ ,$$

and, for all $s \le t$, $t \in [r_j, r_{j+1})$, $j < \kappa$,

$$\mathbb{E}_s\Big[\int_t^{\tau^{\ell}_j} |(Z^{\Re}_u)^{\cdot \ell}|^2 \mathrm{d}u\Big] \le C_L \mathbb{E}_s\Big[\max_{r \in \Re} |\xi_r|^2 + \int_0^T |\eta_u|^2 \mathrm{d}u\Big] \ ,$$

where

$$\tau^{\ell}_j = \inf\{r \in \Re \mid r \ge r_{j+1}, \ \mathbf{1}_{Q^{\ell}_r} \ne 0\} \wedge T \ , \ j \le \kappa - 1 \ , \ \ell \le d \ . \tag{7.5}$$

Proof. 1. Since $\widetilde{Y}^{\Re} \in \mathcal{S}^2(\mathbb{R}^d)$, applying Itô's formula to $|\widetilde{Y}^{\Re}|^2$ on $[r_j, r_{j+1})$, implies

$$\mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2 + \int_t^{r_{j+1}} |Z_u^{\Re}|^2 \mathrm{d}u\Big] = \mathbb{E}_s\Big[|Y_{r_{j+1}}^{\Re}|^2 + 2\int_t^{r_{j+1}} \langle \widetilde{Y}_u^{\Re}, \hat{f}(u)\rangle \mathrm{d}u\Big] ,$$

for all $s \le t \in [r_j, r_{j+1})$, $j < \kappa$.

Fix $\alpha > 1$ to be chosen later on. Combining Cauchy-Schwartz inequality and (7.2) with the inequality $ab \le \alpha a^2 + b^2/\alpha$, $\alpha > 0$, we compute that, for all $s \le t$,

$$\begin{aligned}\mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2 + \int_t^{r_{j+1}} |Z_u^{\Re}|^2 \mathrm{d}u\Big] \quad &\le \quad \mathbb{E}_s\Big[|Y_{r_{j+1}}^{\Re}|^2 + \alpha C_L \int_t^{r_{j+1}} |\widetilde{Y}_u^{\Re}|^2 \mathrm{d}u \\ &\quad + \frac{C_L}{\alpha}\int_t^{r_{j+1}} (|Z_u^{\Re}|^2 + |\eta_u|^2)\mathrm{d}u\Big] .\end{aligned}$$

Taking α large enough such that $C_L/\alpha \le 1/2$, we obtain

$$\mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2 + \frac{1}{2}\int_t^{r_{j+1}} |Z_u^{\Re}|^2 \mathrm{d}u\Big] \le \mathbb{E}_s\Big[|Y_{r_{j+1}}^{\Re}|^2 + C_L \int_t^{r_{j+1}} \Big(|\widetilde{Y}_u^{\Re}|^2 + |\eta_u|^2\Big)\, \mathrm{d}u\Big] .$$

Using Gronwall's Lemma in the last inequality, we then get

$$\mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2 + \frac{1}{2}\int_t^{r_{j+1}} |Z_u^{\Re}|^2 \mathrm{d}u\Big] \le e^{C_L|r_{j+1}-t|}\mathbb{E}_s\Big[|Y_{r_{j+1}}^{\Re}|^2 + C_L \int_t^{r_{j+1}} |\eta_u|^2 \mathrm{d}u\Big] , \quad (7.6)$$

for all $s \le t \in [r_j, r_{j+1})$.

2. It follows easily from (7.6), (7.3), (**A**0) and an induction argument that

$$\sup_{t\in[s,T]} \mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2 + |Y_t^{\Re}|^2\Big] \le C_L\mathbb{E}_s\Big[|\widetilde{Y}_T^{\Re}|^2 + \kappa \max_{r\in\Re} |\xi_r|^2 + \int_s^T |\eta_u|^2 \mathrm{d}u\Big] \quad (7.7)$$

for all $s \le T$.

Moreover, (7.6) applied to $t = r_j$ and $s \le r_j$ reads, recall (7.4),

$$\mathbb{E}_s\Bigg[|\widetilde{Y}_{r_j}^{\Re}|^2 + \frac{1}{2}\int_{r_j}^{r_{j+1}} |Z_u^{\Re}|^2 \mathrm{d}u\Bigg] \le (1 + C_L|\Re|)\mathbb{E}_s\Bigg[|\widetilde{Y}_{r_{j+1}}^{\Re}|^2 + |\xi_{r_{j+1}}|^2 + C_L \int_{r_j}^{r_{j+1}} |\eta_u|^2 \mathrm{d}u\Bigg] ,$$

for $j < \kappa$.

Summing up in this inequality and using (7.7), we obtain

$$\mathbb{E}_s\Big[\int_t^T |Z_u^{\Re}|^2 \mathrm{d}u\Big] \quad \le \quad C_L\mathbb{E}_s\Big[|\widetilde{Y}_T^{\Re}|^2 + \kappa \max_{r\in\Re} |\xi_r|^2 + \int_t^T |\eta_u|^2 \mathrm{d}u\Big] \ , \ s \le t \le T \ ,$$

which concludes the proof of the first claim.

3. We now turn to the case where $(\mathbf{A}f)$ hold. Recalling (7.1) and applying Itô's formula to $|(\widetilde{Y}^{\Re})^{\ell}|^2$ on $[r_j, r_{j+1})$, we get

$$\begin{aligned}|(\widetilde{Y}_t^{\Re})^{\ell}|^2 + \int_t^{r_{j+1}} |(Z_u^{\Re})^{\cdot\ell}|^2 \mathrm{d}u \quad &= \quad |(Y_{r_{j+1}}^{\Re})^{\ell}|^2 - 2\int_t^{r_{j+1}} (\widetilde{Y}_u^{\Re})^{\ell}((Z_u^{\Re})^{\cdot\ell})'\mathrm{d}W_u \\ &\quad +2\int_t^{r_{j+1}} (\widetilde{Y}_u^{\Re})^{\ell} \hat{f}^{\ell}(u)\mathrm{d}u \ , \qquad (7.8)\end{aligned}$$

for all $s \le t \in [r_j, r_{j+1})$, $j < \kappa$.
Recall the definition of τ_q^ℓ. On $\{\tau_q^\ell = r_{q+1}\}$, we obviously have

$$|(\widetilde{Y}_t^{\Re})^\ell|^2 + \int_t^{\tau_j^\ell} |(Z_u^{\Re})^{\cdot\ell}|^2 \mathrm{d}u \le |\xi_{\tau_j^\ell}|^2 - 2\int_t^{\tau_j^\ell} (\widetilde{Y}_u^{\Re})^\ell ((Z_u^{\Re})^{\cdot\ell})' \mathrm{d}W_u$$
$$+2\int_t^{\tau_j^\ell} (\widetilde{Y}_u^{\Re})^\ell \hat{f}^\ell(u)\mathrm{d}u \ .$$

On $\{\tau_q^\ell > r_{q+1}\}$, we denote by θ_q^ℓ the random index such that $r_{\theta_q^\ell} = \tau_q^\ell$. Summing up from q to θ_q^ℓ in (7.8) applied to $t = r_{q+1}$, we retrieve the last inequality.
Arguing as in step 1, recall $(\mathbf{A}f)$, we then obtain

$$\mathbb{E}_s\left[|(\widetilde{Y}_t^{\Re})^\ell|^2 + \frac{1}{2}\int_t^{\tau_q^\ell} |(Z_u^{\Re})^{\cdot\ell}|^2 \mathrm{d}u\right] \le C_L \mathbb{E}_s\left[|\xi_{\tau_q^\ell}|^2 + \int_t^{\tau_q^\ell}\Big(|\widetilde{Y}_u^{\Re}|^2 + |\eta_u|^2\Big)\,\mathrm{d}u\right], \quad (7.9)$$

for all $s \le t \in [r_q, r_{q+1})$, $q < \kappa$.
Summing up on ℓ in the last inequality, we get

$$\mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2\Big] \le C_L \mathbb{E}_s\left[\max_{r\in\Re} |\xi_r|^2 + \int_t^T (|\widetilde{Y}_u^{\Re}|^2 + |\eta_u|^2)\mathrm{d}u\right] \ , \ s \le t \in [r_j, T] \ .$$

Using Gronwall's Lemma, we then have

$$\sup_{t\in[s,T]} \mathbb{E}_s\Big[|\widetilde{Y}_t^{\Re}|^2\Big] \le C_L \mathbb{E}_s\left[\max_{r\in\Re} |\xi_r|^2 + \int_0^T |\eta_u|^2 \mathrm{d}u\right], \ s \le T \ . \quad (7.10)$$

Combining this inequality with (7.9), we also get

$$\mathbb{E}_s\left[\int_t^{\tau_j^\ell} |(Z_u^{\Re})^{\cdot\ell}|^2 \mathrm{d}u\right] \le C_L \mathbb{E}_s\left[\max_{r\in\Re} |\xi_r|^2 + \int_t^T |\eta_u|^2 \mathrm{d}u\right],$$

which concludes the proof. □

Corollary 7.1. *Fix $p \ge 2$ and assume that $\xi \in \mathcal{S}^p(\mathbb{R})$ and $\eta \in L^p(\Omega \times [0,T])$, then when $(\mathbf{A}f)$ holds, we have for all $t \le T$*

$$|\widetilde{Y}_t^{\Re}|^p \quad \le \quad C_L^p \mathbb{E}_t\left[\max_{r\in\Re} |\xi_r|^p + \int_0^T |\eta_u|^p \mathrm{d}u\right] \ .$$

Proof. This follows directly from Jensen's inequality applied to (7.10) with $t = s$. □

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